ISOTROPIC ELASTICITY
INTRODUCTION TO POLYMER AND METAL
SOLID MECHANICS

Aaron Freidenberg

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Chapter 1

Introduction

Before the advent of the personal computer, engineers would perform structural analysis using formulas that are simplified and suitable for hand calculation. Very elegant analytical methods of structural analysis were developed out of necessity and many such methods are still taught today in university and used in industry. Such methods are vital in order to do preliminary “design” calculations, and this author would argue that a deep understanding of classical methods of analysis also give the engineer an intuition about the way forces “flow” through a structure. Most important of all, an intuition about such things will help the reader to understand the topics that will be covered in this text!

Structural analysis work that is done in industry and academia is increasingly being accomplished via structural analysis software. Many undergraduate level courses that deal with such finite element methods (FEM) for structural analysis aim to give the student the confidence (i.e. give the student all of the formulas) that they need so that they can write their own finite element analysis code. In this text, “theory” will be presented in a manner such that FEM itself is the “application.” This text will derive the governing formulas of “elastic isotropic” solid mechanics, whereas courses in FEM take such formulas as a given and focus, instead, on the ways that software implement the formulas, numerically.

note: In this text, FEM refers to “solid” elements (sometimes referred to as “continuum” elements). “Stick” elements (sometimes referred to as “beam” elements) or “shells” are simplifications of “solid” elements.

Consider, for example, the simplest case of isotropic linear infinitesimal elas-
ticity, in FEM. If it’s 2D or 3D, then we need a constitutive equation analogous to \( \sigma = \text{E} \epsilon \) along with a strain-displacement equation analogous to \( \epsilon = \frac{\Delta L}{L} \). The equations \( \sigma = C : \epsilon \) and \( \epsilon_{ij} = \frac{1}{2} \left( \frac{\partial u_i}{\partial X_j} + \frac{\partial u_j}{\partial X_i} \right) \) may look familiar, but in this text they will be derived.

Moreover, this text will cover large (“finite”) strain isotropy as well as provide insight into how more advanced FEM software handle large rigid body rotations and time-dependent problems. The topics covered in this text are essential for engineers in the field of materials research as well as those who use advanced FEM software to its full potential. We will start with “hyperelasticity”, which applies to large strains, large rigid body rotations, and nonlinear (though still elastic) stress-strain relationships. In general, this is the necessary starting point when one has a material with unknown properties. Rubbers actually behave nonlinearly (nonlinear isotropic-elastic behavior is a good starting point) and have large strains.

Then, “hypoelasticity” is considered, which relates stress rates (and therefore incremental stress) to strains and strain rates. Developing the constitutive relationship in hypoelastic form is the preferred method for the most advanced FEM software, such as ABAQUS and LS-DYNA, since they deal with rate and history dependent transient dynamic problems that involve time-stepping algorithms. Lastly, the aforementioned isotropic linear infinitesimal formulation will be discussed. Steel, in the elastic regime, can often be modeled in such a manner.

Before getting into these main ideas, we should introduce tensor notation. In doing so, we’ll also go through some math derivations that will be essential to our formulation of hyperelasticity, linear infinitesimal elasticity, and rate-form constitutive relationships. Tensor math is not a steep learning curve, and it is probably worthwhile to take a bit of time to make sure you understand it. While many of the derivations throughout this text will be complete and quite detailed, from the point of view of most readers, it is important to keep in mind that this text is written in a mechanics style rather than a mathematical style. While the author will do his best to avoid “sloppy” math, the overriding emphasis will always be on the physical interpretation and application of concepts.

This text is only intended to be an introduction to solid mechanics, and therefore only considers “elastic isotropic” materials. The information pro-
vided in this text serves as a prerequisite for topics such as anisotropy, viscoelasticity, and plasticity in solid mechanics.

1.1 Tensor Notation

Throughout this text, we will be working in 3D. The way that the deformations of a body are described will make use of three-component tensors (i.e., vectors) that describe magnitude and direction. We require a vector that has only three components, because we will consider that as a body deforms, any point within the body undergoes a motion that is described by a simple translation, in 3D space. Once we begin looking at strains (and stresses), then we will have to consider the motion of elements, rather than points. Since elements can deform axially in three directions as well as undergo shear, the description of stresses and strains will require the use of higher order tensors (i.e., matrices).

“Tensor” notation (index notation) is essentially a way of keeping track of the components of matrices when performing matrix operations or algebra, without the need to repeatedly draw matrices. In this way, tensors save paper! In addition to this benefit, tensor notation (index notation) lends itself nicely to computer programming. Having said that, the majority of derivations in the later chapters will involve algebra that the reader is probably already familiar with. For example, the basic product between an \(nx1\) vector and a \(1xn\) vector produces an \(nxn\) matrix, while a dot product between the same two vectors would produce a scalar. Similarly, matrix products (denoted with a “·” for reasons that are explained in this chapter), inverses, transposes, and other operations are probably already familiar as well. Basic properties of the operations (e.g., matrix multiplication is associative but not commutative) are also familiar. Thus, it is possible to jump right into the later chapters without studying tensors and only risk becoming “stuck” on the few derivations that operate explicitly on components of matrices.

Tensors (matrices and vectors) will be written in **boldface**. When their components are of interest, alphabetical subscripts (“indices”) will be used, and the boldface removed, since each component of a tensor is merely a scalar. Most vectors will be written as lower-case English letters, while most higher-order tensors will be written using upper-case English letters or Greek letters. Only rectangular (Cartesian) coordinate systems will be
considered.

So, how does tensor math work and in particular, what is “index notation?” Further, what specific tensor operations are going to be most important for us? These are the questions that will be addressed in this chapter.

Consider a typical rectangular coordinate system defined by the unit vectors $e_1, e_2, e_3$ (Fig. 1.1):

![Figure 1.1: Cartesian bases](image)

We know that any vector, $\mathbf{a}$, with components $a_1, a_2, a_3$, can be defined by the units vectors $e_1, e_2, e_3$, as follows:

$$\mathbf{a} = a_1 \mathbf{e}_1 + a_2 \mathbf{e}_2 + a_3 \mathbf{e}_3 = \sum_{i=1}^{3} a_i \mathbf{e}_i \quad (1.1)$$

Our convention (sometimes called the “Einstein Summation Convention”) will be to simply drop the $\sum$ symbol. The subscript (or “index”) will be assumed to be summed. This is one of the important ideas of “tensor notation” (or “index notation”).

**note:** $i$ goes from 1 to 2 in 2D and 1 to 3 in 3D

The dot product between two vectors can be written in index notation:

$$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + a_3 b_3 = a_i b_i \quad (1.2)$$
1.2. **KRONECKER DELTA**

We know that the multiplication of a matrix and a vector results in a vector. The components of this vector can be expressed as follows:

\[
\begin{bmatrix}
  x_1 \\
  x_2 \\
  x_3
\end{bmatrix} =
\begin{bmatrix}
  A_{11} & A_{12} & A_{13} \\
  A_{21} & A_{22} & A_{23} \\
  A_{31} & A_{32} & A_{33}
\end{bmatrix}
\begin{bmatrix}
  b_1 \\
  b_2 \\
  b_3
\end{bmatrix}
\implies x_i = A_{ij}b_j \quad \text{(we'll derive soon)} \quad (1.3)
\]

In eq. (1.3), we are summing over “\( j \)” only, because the practice is to only sum over indices that are “repeated” within a given term in an expression. Another way to think about it, which will always work in this text unless otherwise stated, is to consider that “repeated indices” appear on only one side of the equation, which indicates that they should be summed. The “free index,” which in eq. (1.3) is “\( i \),” appears on both the left-hand side and the right-hand side of the equation (it also only appears one time in any given term) and therefore we know not to sum over “\( i \).”

---

**note:** In eq. (1.3), “\( i \)” is a free index while “\( j \)” is a repeated/dummy index. Repeated indices summed while free indices are not.

---

1.2 **Kronecker Delta**

The “Kronecker Delta”, \( \delta_{ij} \), is a tool that we’ll be using throughout this text. It is simply defined as follows:

\[
\delta_{ij} = \begin{cases} 
1 & \text{if } i = j, \\
0 & \text{if } i \neq j,
\end{cases} \quad (1.4)
\]

The usefulness of the Kronecker Delta lies in its ability to transform indices:

\[
\delta_{ik}a_k = \delta_{i1}a_1 + \delta_{i2}a_2 + \delta_{i3}a_3
\]

- If \( i = 1 \), \( \delta_{ik}a_k = a_1 \)
- If \( i = 2 \), \( \delta_{ik}a_k = a_2 \)
- If \( i = 3 \), \( \delta_{ik}a_k = a_3 \)

\[
\therefore \delta_{ik}a_k = a_i \quad \text{and} \quad \delta_{ik}A_{kj} = A_{ij}
\]

These, and similar ideas involving the Kronecker Delta, will be used extensively in later derivations \((1.5)\)
CHAPTER 1. INTRODUCTION

note: \( \mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij} \)
In other words, the dot product of a unit vector along the “i” axis and a unit vector along the “j” axis is equal to zero, unless \( i \) and \( j \) are equal, in which case the dot product would be 1.

1.3 2nd Order Tensor Transformations

We already know:

\[ \mathbf{a} = a_i \mathbf{e}_i \quad (1.6) \]

Similarly, we need to be able to express a higher order matrix, using tensor notation:

\[ \mathbf{A} = A_{ij} \mathbf{e}_i \mathbf{e}_j \quad (1.7) \]

\( \mathbf{e}_i \mathbf{e}_j \) is sometimes written: \( \mathbf{e}_i \otimes \mathbf{e}_j \), where “\( \otimes \)” denotes the “dyadic” or “tensor” product

\[
\begin{align*}
\text{e.x. } \mathbf{e}_1 \otimes \mathbf{e}_2 &= \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\
\text{As opposed to: } \mathbf{e}_1 \cdot \mathbf{e}_2 &= \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = 0
\end{align*}
\]

\( \mathbf{A} = \text{sum of } 3 \times 3 \text{ matrices } = 3 \times 3 \text{ matrix} \)

\[
\mathbf{A} = A_{ij} \mathbf{e}_i \otimes \mathbf{e}_j = A_{11} \mathbf{e}_1 \otimes \mathbf{e}_1 + A_{12} \mathbf{e}_1 \otimes \mathbf{e}_2 + A_{13} \mathbf{e}_1 \otimes \mathbf{e}_3 \\
+ A_{21} \mathbf{e}_2 \otimes \mathbf{e}_1 + A_{22} \mathbf{e}_2 \otimes \mathbf{e}_2 + A_{23} \mathbf{e}_2 \otimes \mathbf{e}_3 \\
+ A_{31} \mathbf{e}_3 \otimes \mathbf{e}_1 + A_{32} \mathbf{e}_3 \otimes \mathbf{e}_2 + A_{33} \mathbf{e}_3 \otimes \mathbf{e}_3
\]

Recall eq. (1.3), which states that \( x_i = A_{ij} b_j \), where \( \mathbf{A} \) is a 3x3 matrix, \( \mathbf{b} \) is a vector, and \( \mathbf{x} \) is the solution to the product between \( \mathbf{A} \) and \( \mathbf{b} \). To see why this is so, we can consider the product between \( \mathbf{A} \) and the basis vector \( \mathbf{e}_k \), and then consider the product between \( \mathbf{A} \) and an arbitrary vector \( \mathbf{b} \). Before we do this, it should be mentioned that products between tensors and vectors (producing a vector) as well as products between two tensors (producing a tensor) will directly make use of the “dot product,” even though the “dot product” operator is sometimes thought of as a “scalar product.” The
“dot product” treatment used consistently throughout this text, although mathematically a bit “sloppy,” enables one to use index notation to derive quantities in a very direct manner.

The “dot product” will be used in this text to signify the tensor product between a tensor and a vector or the tensor product between two tensors. This is consistent with most of the literature in solid mechanics. A small minority of authors, however, consider this to be “sloppy” math and insist, instead, that the dot product between tensors produce a scalar value. This distinction is very important since tensor products, which produce tensors, are prevalent in solid mechanics and this text will heavily use the “dot” convention when deriving important identities and quantities. Authors that use a different convention would use a very different approach for derivation, in particular where index notation is used.

\[
A \cdot e_k = A_{ij} e_i \otimes e_j \cdot e_k = A_{ij} e_i \delta_{jk} = (A_{ij} \delta_{jk}) e_i = A_{ik} e_i \quad (1.8)
\]

Note the use of the Kronecker delta in simplifying the expression in eq. (1.8).

Similarly,

\[
A \cdot b = A_{ij} e_i e_j \cdot b_k e_k = A_{ij} e_i b_k \delta_{jk} = A_{ik} b_k e_i \quad \text{or} \quad A \cdot b = A_{ij} b_j e_i \quad (1.9)
\]

Again, the solution is a vector; this time with components \( x_i = A_{ij} b_j \), as expected (eq. (1.3)).

Note: Eq. (1.9) is standard matrix multiplication. We had to use a “dot” product to get the tensor algebra to work as a typical matrix - vector multiplication (we’ll see later that \( A \cdot B \) also results in the standard matrix - matrix product). This is consistent with most of the literature in solid mechanics.

Note: Remember, in the above derivations, only if \( j = k \) is \( \delta_{jk} \) non-zero. This was explained in eq (1.5). Note how this simplified the derivations. This is very common in tensor algebra.

So, consider eq. (1.8). Let’s “pick out” a particular component of this tensor, as follows:

Multiplying both sides by \( e_j \rightarrow e_j \cdot A \cdot e_k = e_j \cdot A_{ik} e_i = \delta_{ij} A_{ik} = A_{jk} \)

\[\therefore\]
$A_{jk} = \mathbf{e}_j \cdot \mathbf{A} \cdot \mathbf{e}_k$ or, re-written:

$$A_{ij} = \mathbf{e}_i \cdot \mathbf{A} \cdot \mathbf{e}_j$$  \hspace{1cm} (1.10)

note: The LHS of eq. (1.10) is only one term. I.e. this is how you pick out a single term of a matrix.

note: For $1^{st}$ order: $a_i = \mathbf{a} \cdot \mathbf{e}_i$, where $a_i$ gives the component of $\mathbf{a}$ in the "$i^{th}$" direction.

Consider the following coordinate system (Fig. 1.2):

Figure 1.2: Coordinate transformation

$$\mathbf{e}_i' = Q \cdot \mathbf{e}_i \; ; \; \mathbf{Q} = \text{rotation matrix} = Q_{mn} \mathbf{e}_m \otimes \mathbf{e}_n$$

we know: $\mathbf{e}_i' = Q_{mn} \mathbf{e}_m \otimes \mathbf{e}_n \cdot \mathbf{e}_j = Q_{mi} \mathbf{e}_m \rightarrow \mathbf{e}_i' \cdot \mathbf{e}_n = Q_{mi} \mathbf{e}_m \cdot \mathbf{e}_n = Q_{ni}$

$\rightarrow Q_{ni} = \mathbf{e}_i' \cdot \mathbf{e}_n$ (terms [components] of a rotation matrix)

To be more useful, we need to show that $a_j = Q_{ji}a_i'$.

By definition, $\mathbf{a} = a_i \mathbf{e}_i = a_i' \mathbf{e}_i$

Multiply both sides by $\mathbf{e}_j : a_i \mathbf{e}_i \cdot \mathbf{e}_j = a_i' \mathbf{e}_i' \cdot \mathbf{e}_j$

And, since $Q_{ji} = \mathbf{e}_j \cdot \mathbf{e}_i' = \mathbf{e}_i' \cdot \mathbf{e}_j$ (this should be obvious):
1.3. 2ND ORDER TENSOR TRANSFORMATIONS

\[ a_j = Q_{ji} a'_i \]  \hspace{1cm} (1.11)

\textbf{note: } \( Q \cdot Q^{-1} = Q^{-1} \cdot Q = I \) (this is true of any tensor)

\( Q \) is an “orthogonal” tensor. A particular identity of an orthogonal tensor can be written as follows:

\[ (Q \cdot u) \cdot (Q \cdot v) = u \cdot v \]  \hspace{1cm} (1.12)

For an orthogonal tensor, \( Q \) (obeys eq. (1.12)), it can be shown that \( Q^T = Q^{-1} \) (pf: [2])

\[ \therefore \]

\[ Q \cdot Q^T = Q^T \cdot Q = I \]

where \( I = \delta_{ij} e_i \otimes e_j \)

\[ = e_i e_i = e_1 e_1 + e_2 e_2 + e_3 e_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \]

From eq. (1.11), we know that \( a = Q \cdot a' \)

\[ Q^T \cdot a = \underbrace{Q^T \cdot Q \cdot a'}_{I} \]

\[ a' = Q^T \cdot a \]

\[ a'_i = (Q^T)_{ij} a_j \]

\[ a'_i = Q_{ji} a_j \]  \hspace{1cm} (1.13)

In eq. (1.13), \( Q_{ji} = e_j \cdot e'_i \) (assuming we know the unit vectors defining the original and rotated coordinate systems).

What about transforming a tensor?

\[ A = A_{ij} e_i \otimes e_j \]
\[ A = A_{ij} e_i \otimes e_j \]

\[ e'_k \cdot A \cdot e'_l = A_{ij} (e'_k \cdot e'_1) (e_j \cdot e'_1) \]

\[ A'_{kl} = Q_{ik} Q_{jl} \]
The following will be simply stated, and then proved:

\[ A'_{kl} = A_{ij}Q_{ik}Q_{jl} = Q_{ik}A_{ij}Q_{jl} \rightarrow A' = Q^T \cdot A \cdot Q \]  

(1.14)

Tensor product practice and quick proof of eq. (1.14):

\[ A \cdot Q = A_{ij}e_je_j \cdot Q_{kl}e_ke_k = A_{ij}Q_{kl}\delta_{jk}e_ie_i = A_{ij}Q_{jl}e_ie_i \]

Now taking \( Q \) to be \( Q_{tk}e_te_k \), we find:

\[ Q^T \cdot (A \cdot Q) = Q_{tk}e_ke_k \cdot A_{ij}Q_{jl}e_ie_i \]

Thus, \( Q^T \cdot A \cdot Q = Q_{tk}A_{ij}Q_{jl}\delta_{ti}e_ke_k = Q_{ik}A_{ij}Q_{jl}e_ke_k \)

Setting \( A' = Q^T \cdot A \cdot Q \), we can see that the components are \( A'_{kl} = Q_{ik}A_{ij}Q_{jl} \)

- i.e. the desired result (eq. (1.14))

\[ tr(A) = A_{ii} \]
\[ tr(A \cdot B) = tr(C) = C_{ii} \]
\[ C_{ij} = A_{ik}B_{kj} \neq A_{ik}B_{jk} \text{ unless the tensor is symmetric} \]
\[ tr(A \cdot B) = C_{ii} = A_{ik}B_{ki} \]

\begin{itemize}
  \item \textbf{1.4 Trace, Scalar Product, Eigenvalues}
  \item "Trace" is a particular operator that, when "applied" to a 2\textsuperscript{nd} order (or higher) tensor, sums the diagonal components. Strictly-speaking, the definition of "trace" can be taken as \( tr(u \otimes v) = u \cdot v \)
  \item \( tr(A) = A_{ii} \)
  \item \( tr(A \cdot B) = tr(C) = C_{ii} \)
  \item \( C_{ij} = A_{ik}B_{kj} \neq A_{ik}B_{jk} \text{ unless the tensor is symmetric} \)
  \item \( tr(A \cdot B) = C_{ii} = A_{ik}B_{ki} \)
\end{itemize}

\begin{itemize}
  \item note: We put a “dot” to get the tensor algebra to work. It’s really just a standard matrix product. If ever a product is written \( AB \) or \( Ab \), in this text, it is probably a typo! Most of the literature in solid mechanics uses the same notation as in this text. Some authors omit the “dot” (e.x. \( A \cdot B = AB \)) except when using index notation. Some authors take \( A \cdot B \) to be the scalar product, which is a completely different definition, as opposed to a difference merely in notation. Fortunately, authors that take \( A \cdot B \) to be the scalar product are a small minority in solid mechanics.
\end{itemize}
The "scalar product" of tensors is analogous to the "dot product" of vectors. In this text, the scalar product is denoted by the operator "::" and is defined as follows:

\[ \mathbf{A} : \mathbf{B} = tr(\mathbf{A}^T \cdot \mathbf{B}) = tr(\mathbf{A} \cdot \mathbf{B}^T) = A_{ik}B_{ik} \]

We can arrive at the same result in a different manner:

\[ \mathbf{A} : \mathbf{B} = A_{ij}e_i e_j : B_{lk}e_l e_k = A_{ij}B_{lk}\delta_{il}\delta_{jk} = A_{ik}B_{ik} \]

**Note:** Since the scalar product of two tensors is analogous to the "dot product" of two vectors, some authors define the scalar product between \( \mathbf{A} \) and \( \mathbf{B} \) as \( \mathbf{A} \cdot \mathbf{B} \) [15]. In other words, these authors define \( \mathbf{A} \cdot \mathbf{B} = A_{ik}B_{ik} \). Using such a definition, \( \mathbf{A} \cdot \mathbf{B} \), which still must be equal to \( A_{ij}e_i e_j \cdot B_{lk}e_l e_k \), can no longer be expressed as \( A_{ij}B_{lk}\delta_{il}\delta_{jk} e_i e_l \), as was done previously. Derivations in solid mechanics that use the simple manipulation shown in the last term would have to be done using some other approach that is likely more cumbersome.

An eigenvector, \( \mathbf{n} \), of a tensor or a matrix has the following special property: when \( \mathbf{n} \) is multiplied by the matrix, the result is a new vector that has the same direction as \( \mathbf{n} \). The amount by which the magnitude of the vector has changed is the value of the eigenvalue (the eigenvalue that corresponds to the eigenvector \( \mathbf{n} \)).

\[ i.e. \ (\mathbf{A} - \lambda \mathbf{I}) \cdot \mathbf{n} = 0 \]

\( \lambda \) are the eigenvalue solutions and \( \mathbf{n} \) are the corresponding eigenvectors

Any tensor product can be expressed in terms of invariants (or eigenvalues) i.e.

\[ \epsilon = [\lambda_1n_1^1n_j^1 + \lambda_2n_i^2n_j^2 + \lambda_3n_i^3n_j^3]e_i \otimes e_j \] (skipped work)

or

\[ \epsilon = \sum_{a=1}^3 \lambda_a \mathbf{n}_a \otimes \mathbf{n}_a \]

Characteristic equation and Cayley-Hamilton Theorem:

One would typically solve for the eigenvalues from a "characteristic equa-
tion” of the form \( \lambda^3 - I_A \lambda^2 + II_A \lambda - III_A = 0 \) (for a 3x3 matrix), where \( I_A, II_A, III_A \) are coefficients that depend on the values within the matrix, \( A \). These coefficients, \( I_A, II_A, III_A \), are more commonly called invariants.

\[
\begin{align*}
I_A &= \lambda_1 + \lambda_2 + \lambda_3 \\
II_A &= \lambda_1 \lambda_2 + \lambda_2 \lambda_3 + \lambda_3 \lambda_1 \\
III_A &= \lambda_1 \lambda_2 \lambda_3
\end{align*}
\]

We’ll see some alternative expressions for \( I_A, II_A, III_A \) later, when we get to “hyperelasticity.”

Note: The invariants of a matrix are the same regardless of coordinate system (as is the trace).

The matrix also satisfies its own characteristic equation (this is known as the Cayley-Hamilton Theorem).

\[
\begin{align*}
A^3 - I_A A^2 + II_A A - III_A I &= 0 \quad (1.15) \\
A^{-1}(A^3 - I_A A^2 + II_A A) &= III_A IA^{-1} \\
A^2 - I_A A + II_A I &= III_A A^{-1}
\end{align*}
\]

\[
\frac{1}{III_A}(A^2 - I_A A + II_A I) = A^{-1} \quad (1.16)
\]

Either eq. (1.15) or eq. (1.16) are often used in derivations of hyperelasticity later on.
Chapter 2

Strain

Strain is a measure of deformation that has an intuitive physical meaning for engineers. Specifically, axial strain, for example, is a measure of the deformation of a structure (or element within a structure) with respect to the structure’s length. Thus, strain is a ratio of lengths and is a unit-less quantity. While strain can come in the form of extension, shortening, or shear strain, it is sometimes overlooked that strain also has direction. This change in direction, under, for example, rigid body rotation, creates some interesting complications in the finite element world.

In the finite element world and, accordingly, in solid mechanics (“solid” or “continuum” elements), we essentially calculate the strain at every point within a structure. Even for very simple geometries subjected to symmetric loads, some of the points (elements) within a structure will undergo rigid body rotations. Such rigid body motions present a complication: the elements comprising the structure need to maintain a consistent frame of reference. The need for a consistent frame of reference is even more apparent when one considers impact type behavior that involves multiple bodies.

If the reader is familiar with the idea of “geometric nonlinearity,” then following analogy may be useful. Consider the manner in which structural engineers perform the preliminary design of the beams and columns of tall buildings. They often design for maximum flexibility, using efficient analytical methods. Where gravity and lateral loads act simultaneously on a building, these engineers know that the force demands in the structure from the gravity loads should be calculated after the force demands from the lateral loads are determined. The lateral loads cause the building to sway, and
the presence of rigid body rotation of the columns is important to consider prior to calculating the force demands from the gravity loads. Similarly, in FEM, there is a certain order to the treatment of rigid body rotation in the FEM algorithms, which we will see when we get to the chapter on “Rate-Form Constitutive Expressions.” It is important to recognize this now, however, because we will begin discussion of strain and many different measures of strain will be presented. Different strain measures are used in FEM, depending on the order in which rigid body rotations are considered.

Since the analytical methodology for building design was mentioned, now is a good time to remind the reader that the kinds of FEM analysis that this text considers are fundamentally different from analytical methods of design and analysis. ABAQUS/Explicit and LS-DYNA are examples of the kinds of FEM software considered in this text. Here, geometric nonlinearity is not an issue. Rather, we are free to model structures of complex geometries subject to any loading conditions we wish. In addition, the behavior of “solid” elements are governed by continuum-type mechanics, and so “moments” will not be considered. To really develop the framework used by FEM codes, though, we need to consider stress, as well as constitutive expressions relating stress and strain, including “hyperelasticity.” This chapter only provides an overview of the various strain tensors that will be used later on in this text.

Consider a body that undergoes deformation as well as rigid body rotations (Fig. 2.1). In particular, we’ll consider a vector that is initially \( d\mathbf{X} \), and well will track this vector as it becomes \( d\mathbf{x} \).

Define:

\[
dx = \mathbf{F} \cdot d\mathbf{X} \tag{2.1}
\]

In eq. (2.1), \( \mathbf{F} \) is a second-order tensor known as the “deformation gradient”

We can see from fig. 2.1:

\[
\begin{align*}
\mathbf{x} + d\mathbf{x} &= \mathbf{X} + d\mathbf{X} + \mathbf{u}(\mathbf{X} + d\mathbf{X}, t) \\
\rightarrow d\mathbf{x} &= d\mathbf{X} + \mathbf{u}(\mathbf{X} + d\mathbf{X}, t) - \mathbf{u}(\mathbf{X}, t)
\end{align*}
\]

where \( \mathbf{u}(\mathbf{X}, t) = \mathbf{x} - \mathbf{X} \)

In index notation, \( dx_i = dX_i + u_i(X_j + dX_j, t) - u_i(X_j, t) \)

where \( u_i(X_j + dX_j, t) = u_i(X_j, t) + \frac{\partial u_i}{\partial X_j} dX_j \), which comes from a Taylor
2.1. STRAIN TENSORS

Expansion and neglecting higher order terms
\( i.e. f(x + \Delta x) \approx f(x) + \frac{df(x)}{dx} \Delta x + \frac{1}{2} \frac{d^2f(x)}{dx^2} \Delta x^2 + ... \)

So, \( dx_i = dX_i + \frac{\partial u_i}{\partial X_j} dX_j \) or \( dx = dX + dX \cdot (\nabla u) \)

Note the “Nabla” or “gradient” symbol \( \nabla \) that is a commonly used operator

So, \( dx = (I + u\nabla) \cdot dX \)

2.1 Strain Tensors

Recall from eq. (2.1), that \( dx = F \cdot dX \)

Since \( dx = (I + u\nabla) \cdot dX \), \( F = I + u\nabla \)

or, in index notation:

\( F_{ij} = \delta_{ij} + \frac{\partial u_i}{\partial X_j} = \frac{\partial x_i}{\partial X_j} \), since \( du = [(x + dx) - (X + dX)] - (x - X) = dx - dX \)

and \( \frac{\partial x_i}{\partial X_j} = \delta_{ij} \), by definition.
In matrix form:

\[
\mathbf{u} \nabla = \begin{bmatrix}
\frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \frac{\partial u_1}{\partial x_3} \\
\frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} & \frac{\partial u_2}{\partial x_3} \\
\frac{\partial u_3}{\partial x_1} & \frac{\partial u_3}{\partial x_2} & \frac{\partial u_3}{\partial x_3}
\end{bmatrix}
\]  

(2.2)

\[\mathbf{F} = \text{“deformation gradient”}:
\begin{bmatrix}
\frac{\partial x_1}{\partial X_1} & \frac{\partial x_1}{\partial X_2} & \frac{\partial x_1}{\partial X_3} \\
\frac{\partial x_2}{\partial X_1} & \frac{\partial x_2}{\partial X_2} & \frac{\partial x_2}{\partial X_3} \\
\frac{\partial x_3}{\partial X_1} & \frac{\partial x_3}{\partial X_2} & \frac{\partial x_3}{\partial X_3}
\end{bmatrix}
\]  

(2.3)

\(\mathbf{F}\) is sometimes called the “stretch” tensor.

Now, consider how the length of any element or fiber within the continuum may change under deformation. To find such length “magnitudes” we can take dot products as follows:

\[dS^2 = d\mathbf{X} \cdot d\mathbf{X} \quad (dS = \text{length } d\mathbf{X})\]
\[ds^2 = dx \cdot dx \quad (ds = \text{length } dx)\]
\[d\mathbf{x} = \mathbf{F} \cdot d\mathbf{X} = d\mathbf{X} \cdot \mathbf{F}^T\]

**note:** you can’t do this transpose manipulation as easily if multiplying two tensors, but it works for two vectors or a vector and a tensor (use indices to easily prove)

\[ds^2 = d\mathbf{X} \cdot \mathbf{F}^T \cdot \mathbf{F} \cdot d\mathbf{X}\]

**note:** \(\mathbf{F}^T \cdot \mathbf{F} = \mathbf{C} = \text{“Right C - G” (Cauchy - Greene) deformation tensor. The reason for this name will become clear once we begin discussion our on “polar decomposition” theory.}\)

\[
\frac{ds^2 - dS^2}{dS^2} = \frac{d\mathbf{X} \cdot (\mathbf{F}^T \cdot \mathbf{F} - \mathbf{I}) \cdot d\mathbf{X}}{d\mathbf{X} \cdot d\mathbf{X}} = 2\mathbf{E}
\]

(2.4)

where \(\mathbf{E} = \text{Lagrangian Strain Tensor} = \frac{1}{2}(\mathbf{F}^T \cdot \mathbf{F} - \mathbf{I})\)

or

\[\mathbf{E} = \frac{1}{2}(\mathbf{C} - \mathbf{I})\]

(2.5)
2.1. STRAIN TENSORS

\( E \cdot n \) = strain vector on a plane whose normal vector is \( n \) (the actual location in space is specified within \( E \))

\( n \cdot E \cdot n \) = strain (scalar) in direction \( n \)

**Note:**
From eq. (2.4), \( E_{nn} = \frac{ds^2 - dS^2}{2dS^2} = \frac{(ds-dS)(ds+dS)}{2dS^2} \)

\( E_{nn} \approx \frac{(ds-dS)2dS}{2dS^2} = \frac{ds-dS}{dS} = \epsilon_{nn} \)

i.e. \( \epsilon = \frac{\Delta L}{L} \)

What about \( m \cdot E \cdot n \)?

E.x.
Suppose \( n \) and \( m \) are orthogonal in the undeformed configuration.
Since \( E \cdot n \) is the strain vector on a plane whose unit normal is \( n \),
\( m \cdot E \cdot n \) is the component of \( E \cdot n \) in direction \( m \).
i.e. if \( m \) is in the plane of interest (orthogonal to \( n \)), then \( m \cdot E \cdot n \) is the shear strain

\[
\begin{align*}
\vec{dX}_2 &= ds_2 \vec{n} \\
\vec{dX}_1 &= ds_1 \vec{m}
\end{align*}
\]

**Figure 2.2: Initial configuration and deformed configuration**

We know that: \( d\vec{x}_1 = F \cdot d\vec{X}_1 \); \( d\vec{x}_2 = F \cdot d\vec{X}_2 \)

So, \( d\vec{x}_1 \cdot d\vec{x}_2 - d\vec{X}_1 \cdot d\vec{X}_2 = F \cdot d\vec{X}_1 \cdot F \cdot d\vec{X}_2 - d\vec{X}_1 \cdot d\vec{X}_2 \)

From eq. (2.4) and recalling that \( F \cdot d\vec{X} = d\vec{X} \cdot F^T \), we get:
\( d\vec{x}_1 \cdot d\vec{x}_2 - d\vec{X}_1 \cdot d\vec{X}_2 = 2d\vec{X}_1 \cdot E \cdot d\vec{X}_2 \)

\[
\begin{align*}
ds_1 ds_2 \hat{m} \cdot \hat{n} &= -2dS_1 dS_2 \hat{m} \cdot E \cdot \hat{n} \\
&= E_{mn}, \text{ where } \hat{m} \text{ and } \hat{n} \text{ are unit vectors}
\end{align*}
\]
CHAPTER 2. STRAIN

\[ E_{mn} = \frac{1}{2} \frac{ds_1}{dS_1} \frac{ds_2}{dS_2} \cos \alpha \]

where \( \frac{ds_1}{dS_1} \) is the “stretch ratio”

note: As seen in Fig. 2.3, \( \gamma = \frac{\pi}{2} - \alpha \), \( \cos \alpha = \sin \gamma \)

Figure 2.3: Engineering strain, \( \gamma \)

\[ E_{mn} = \frac{1}{2} \frac{ds_1}{dS_1} \frac{ds_2}{dS_2} \sin \gamma \]

Infinitesimal engineering shear strain = \( \epsilon_{xy} = \frac{1}{2} (\gamma_1 + \gamma_2) = \frac{1}{2} \gamma \)

Does our shear strain reduce to this value for infinitesimal deformation?

For infinitesimal deformation, \( \frac{ds_1}{dS_1} = 1 ; \frac{ds_2}{dS_2} = 1 ; \sin \gamma = \gamma \)

\[ E_{mn} = \frac{1}{2} (1)(1)(\gamma) = \frac{1}{2} \]

\[ E_{mn} \approx \epsilon_{mn} \] for linear infinitesimal deformation

We can also see from the following equation (eq. (2.6)) that, in general, \( \epsilon_{ij} = \frac{1}{2} (\frac{\partial u_i}{\partial X_j} + \frac{\partial u_j}{\partial X_i}) \) for linear infinitesimal deformation (higher order terms are neglected).

For large (“finite”) strain:

\[ E_{ij} = \frac{1}{2} \left( \frac{\partial u_i}{\partial X_j} + \frac{\partial u_j}{\partial X_i} + \frac{\partial u_k}{\partial X_i} \frac{\partial u_k}{\partial X_j} \right) \quad (2.6) \]

To prove, first consider: \( F_{ij} = \delta_{ij} + \frac{\partial u_i}{\partial X_j} \rightarrow F = I + u \nabla \)

Now, from eq. (2.4), \( E = \frac{1}{2} (F^T \cdot F - I) = \frac{1}{2} [(I + \nabla u) \cdot (I + u \nabla) - I] \)

\[ = \frac{1}{2} [(I + \nabla u) + \nabla u + (\nabla u) \cdot (u \nabla) - I] = \frac{1}{2} [u \nabla + \nabla u + (\nabla u) \cdot (u \nabla)] - I \]

Recall the definition of \( \nabla u \) from Fig. 2.1, and recall that \( u \nabla \) is the transpose of \( \nabla u \)
2.1. STRAIN TENSORS

Eulerian strain:
Here, “Eulerian strain” is simply referring to a measure of strain that is defined in spatial coordinates. Under rigid body rotation, the Eulerian strain values will change, whereas the Lagrangian strain tensor is invariant to rigid body rotation. In other words, the coordinate system in which \( \mathbf{E} \) is calculated (the “material” coordinate system) rotates with rigid body rotation. The coordinate system in which \( \mathbf{e} \) is calculated (the “spatial” coordinate system) remains constant.

\[
dx = \mathbf{F} \cdot d\mathbf{X} \rightarrow d\mathbf{X} = \mathbf{F}^{-1} \cdot dx = dx \cdot \mathbf{F}^{-T}
\]

Similar to the way that we derived \( \mathbf{E} \), let’s consider the difference in lengths of any particular element, or fiber, within our strain potato, before and after deformation.

\[
ds^2 - dS^2 = dx \cdot dx - dX \cdot dX = dx \cdot dx - dx \cdot \mathbf{F}^{-T} \cdot \mathbf{F}^{-1} \cdot dx = dx \cdot (\mathbf{I} - \mathbf{F}^{-T} \cdot \mathbf{F}^{-1}) \cdot dx
\]

\[
ds^2 - dS^2 = 2dx \cdot \mathbf{e} \cdot dx
\]

where the Eulerian Strain Tensor \( \mathbf{e} = \frac{1}{2} (\mathbf{I} - \mathbf{F}^{-T} \cdot \mathbf{F}^{-1}) \)

or

\[
\mathbf{e} = \frac{1}{2} (\mathbf{I} - \mathbf{B}^{-1}) \quad (2.7)
\]

where \( \mathbf{B} = \) Left C-G Tensor = \( \mathbf{F} \cdot \mathbf{F}^T \)

\( (\mathbf{B}^{-1} = \mathbf{F}^{-T} \cdot \mathbf{F}^{-1}) \), which is easy to prove using indices

\[
d\mathbf{X} \cdot d\mathbf{X} = dx \cdot \underbrace{\mathbf{F}^{-T} \cdot \mathbf{F}^{-1}}_{\mathbf{B}^{-1}} \cdot dx
\]

This is analogous to the previously derived \( dx \cdot dx = d\mathbf{X} \cdot \mathbf{F}^T \cdot \mathbf{F} \cdot d\mathbf{X} \)

Proof comes from:

\[
d\mathbf{X} = \mathbf{F}^{-1} \cdot dx = dx \cdot \mathbf{F}^{-T}
\]

We’ll see the physical meaning of \( \mathbf{B} \) and \( \mathbf{C} \) when we discuss “polar decomposition.”
How is $e$ related to $E$?

$$E = \frac{1}{2} (\mathbf{F}^T \cdot \mathbf{F} - \mathbf{I}) ; e = \frac{1}{2} (\mathbf{I} - \mathbf{F}^{-T} \cdot \mathbf{F}^{-1})$$

$$E = \mathbf{F}^T \cdot \frac{1}{2} (\mathbf{I} - \mathbf{F}^{-T} \cdot \mathbf{F}^{-1}) \cdot \mathbf{F}$$

$$e = \mathbf{F}^{-T} \cdot \frac{1}{2} (\mathbf{F}^T \cdot \mathbf{F} - \mathbf{I}) \cdot \mathbf{F}^{-1}$$

$$\mathbf{B}^{-1} = \mathbf{F}^{-T} \cdot \mathbf{F}^{-1} \longrightarrow B_{ij}^{-1} = F_{ik}^{-T} \cdot F_{kj}^{-1}$$

(To convince yourself that these subscripts are correct, simply write out the matrix multiplication long-hand, summing only the dummy index “k”)

Since $\frac{\partial X_i}{\partial x_j} = \delta_{ij} - \frac{\partial u_i}{\partial x_j} = F_{ij}^{-1}$,

$$B_{ij}^{-1} = (\delta_{ik} - \frac{\partial u_k}{\partial x_i})(\delta_{kj} - \frac{\partial u_k}{\partial x_j}) = \delta_{ij} - \frac{\partial u_i}{\partial x_j} - \frac{\partial u_i}{\partial x_i} + \frac{\partial u_k}{\partial x_i} \frac{\partial u_k}{\partial x_j}$$

$$e = \frac{1}{2} (\mathbf{I} - \mathbf{B}^{-1}) \longrightarrow e_{ij} = \frac{1}{2} (\delta_{ij} - B_{ij}^{-1}) = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{\partial u_k}{\partial x_i} \frac{\partial u_k}{\partial x_j} \right)$$

If only infinitesimal deformation, and so long as no significant rigid body rotations are present, then $E_{ij} \approx e_{ij} \approx \epsilon_{ij} = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)$

(this formula may be familiar from undergrad, for example)

2.2 Volume and Area Change

For completeness, we’ll derive the volume change and area change:

$$dV = dV_0 |\text{det} \mathbf{F}|$$

Derivation starts from the triple scalar product. Cross products and determinants (written in tensor form) is a math topic, so we’ll skip the derivation of volumetric deformation

\[\text{note: } |\text{det} \mathbf{F}| = 1 \longrightarrow dV = dV_0 \longrightarrow \text{“incompressible”}\]

Nanson’s Equation:
Now we know that $dV = (|\text{det} \mathbf{F}|)dV_0$, but what about area?
2.3. POLAR DECOMPOSITION THEORY

If $dS_0$ is the initial area and $dS$ is the final area:
For small volumes, we can say that $dV_0 = dS_0 \mathbf{dX}$ and $dV = dS \mathbf{dx}$.

It was previously shown that $dV = \det \mathbf{F}dV_0$
\[ \therefore dV_0 = dS_0 \mathbf{dX} \text{ and } dV = \det \mathbf{F}dV_0 = dS \mathbf{dx} \rightarrow dV_0 = \frac{dS \mathbf{dx}}{\det \mathbf{F}} \]
Substituting, we get $dS \mathbf{dx} = dS_0 \mathbf{dX} \det \mathbf{F}$
We also know that $d \mathbf{x} = \mathbf{F} \cdot d \mathbf{X}$

\[ \rightarrow dS \mathbf{F}^T \mathbf{dX} = dS_0 \mathbf{dX} \det \mathbf{F} \rightarrow (\det \mathbf{F}dS_0 - \mathbf{F}^TdS) \cdot \mathbf{dX} = 0 \]
\[ \rightarrow dS = dS \mathbf{n} = \det \mathbf{F} \star \mathbf{F}^{-T}dS_0 = \det \mathbf{F} \star \mathbf{F}^{-T} \cdot \mathbf{N}dS_0 \quad (2.10) \]

where $\mathbf{N}$ and $\mathbf{n}$ are the normal vectors to the surface in the respective initial and final configurations.

Eq. (2.10) is called Nanson’s Equation and will be useful later when we look at “true” stress versus “nominal” stress, for example.

2.3 Polar Decomposition Theory

Any matrix can be decomposed into a sum of symmetric and antisymmetric matrices, but $\mathbf{F}$ can be decomposed into a product of two matrices (one symmetric and one orthogonal)
\[ F = V \cdot R = R \cdot U \quad (2.11) \]

\( U \) and \( V \) are called the Right Stretch Tensor and Left Stretch Tensor due to their respective positions (relative to \( R \)) in eq. (2.11).

\( V = V^T \) (symmetric)

\( U = U^T \) (symmetric)

\( R^T = R^{-1} \) (orthogonal)

A proof of the orthogonality of \( R \) is given in Appendix A.3.

\[ C = F^T \cdot F = U \cdot R^T \cdot R \cdot U = U^2 \quad (2.12) \]

\[ B = V^2 \quad (2.13) \]

\( R \) is a rigid body rotation, while \( U \) or \( V \) each stretch and rotate (they each contain both normal and shear deformations, typically)

\( U \) and \( V \) have the same eigenvalues. Eigenvalues do not have an “order,” \textit{per se}, but since \( U \) and \( V \) typically have different eigenvectors, one should be cautious when assuming equivalency of eigenvalues. In the principal directions of \( U \) or \( V \), \( U \) or \( V \) contains no shear deformation (fig. 2.5). We’ll look at actual stresses, later.

To find the principal directions of \( U \) and \( V \), we must solve the eigenvalue problem:

\[ U \cdot n = \lambda n \]

\[ R \cdot U \cdot n = R \cdot \lambda n \] (pictured)

\[ F = V \cdot R \rightarrow V \cdot (R \cdot n) = \lambda (R \cdot n) \]

\[ m_i = R \cdot n_i \]
2.3. **POLAR DECOMPOSITION THEORY**

Figure 2.5: Deformation in principal directions vs. deformation in typical direction

Figure 2.6: The principal directions for pure shear deformation are at 45°
e.x. Polar Decomposition

The deformed equilibrium configuration of a body defined by the deformation mapping:

\[ x_1 = X_1 + 3X_2 , \ x_2 = X_2 , \ x_3 = X_3 \]

Determine:

a) \(F\) and \(C\)

b) Eigenvalues and eigenvectors of \(C\)

c) \(U\) and \(U^{-1}\)

d) \(R\)

e) \(V\)

\[ a) \ F = \begin{bmatrix} \frac{\partial x_1}{\partial X_1} & \frac{\partial x_1}{\partial X_2} & \frac{\partial x_1}{\partial X_3} \\ \frac{\partial x_2}{\partial X_1} & \frac{\partial x_2}{\partial X_2} & \frac{\partial x_2}{\partial X_3} \\ \frac{\partial x_3}{\partial X_1} & \frac{\partial x_3}{\partial X_2} & \frac{\partial x_3}{\partial X_3} \end{bmatrix} = \begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \]

Note: \(F\) will typically be a function of \(X_1, X_2, X_3\) and we'll be interested in the value of \(F\) at a particular point. Here, it doesn't matter.

\[ \text{b) } C \cdot n = \lambda n ; \ det| - \lambda I + C | = 0 \]

\[ \rightarrow \lambda_1 = .092 ; \ \lambda_2 = 1 ; \ \lambda_3 = 10.91 \]

\[ [-\lambda_i I + C]n_i = 0 \]

i.e. for \(\lambda_1\),

\[ .908n_1 + 3n_2 = 0 \]
\[ 3n_1 + 9.908n_2 = 0 \]
\[ .908n_3 = 0 \]

Choose arbitrary \(n_1\); solve for \(n_2\) from either equation; normalize:
2.3. POLAR DECOMPOSITION THEORY

\[
\begin{bmatrix}
  n_1 \\
  n_2 \\
  n_3
\end{bmatrix}
\lambda_1 =
\begin{bmatrix}
  .957 \\
  -.290 \\
  0
\end{bmatrix} ;
\begin{bmatrix}
  n_1 \\
  n_2 \\
  n_3
\end{bmatrix}
\lambda_2 =
\begin{bmatrix}
  0 \\
  1 \\
  0
\end{bmatrix} ;
\begin{bmatrix}
  n_1 \\
  n_2 \\
  n_3
\end{bmatrix}
\lambda_3 =
\begin{bmatrix}
  .290 \\
  .957 \\
  0
\end{bmatrix}
\]

c) \([U]_n = \sqrt{[C]_n} =
\begin{bmatrix}
  \sqrt{\lambda_1} & \sqrt{\lambda_2} & \sqrt{\lambda_3}
\end{bmatrix} =
\begin{bmatrix}
  .303 & 1 & 3.303
\end{bmatrix}
\]

\([U]_e = [\Phi][U]_n[\Phi]^T =
\begin{bmatrix}
  .56 & .83 & 0 \\
  .83 & 3.05 & 0 \\
  0 & 0 & 1
\end{bmatrix} \text{ (symmetric)}
\]

where \([\Phi] =
\begin{bmatrix}
  (n_1)_{\lambda_1} & (n_1)_{\lambda_2} & (n_1)_{\lambda_3} \\
  (n_2)_{\lambda_1} & (n_2)_{\lambda_2} & (n_2)_{\lambda_3} \\
  (n_3)_{\lambda_1} & (n_3)_{\lambda_2} & (n_3)_{\lambda_3}
\end{bmatrix}
\]

It’s important to keep in mind that \(U, E, \) and \(C\) have the same eigenvectors and that these eigenvectors (and any other strain direction or magnitude) are generally dependent on the particular point in space of interest (e.g. \(X_1, X_2, X_3\)). For this simple example, \(F\) is independent of \(X_1, X_2, \) and \(X_3\) (i.e. the deformation is the same everywhere in the body - a “homogenous” deformation), but this would generally not be the case.

The eigenvalues occur in the direction of the eigenvectors, thus, \(n_1 \cdot E \cdot n_1 = \lambda_1 ; n_2 \cdot E \cdot n_2 = \lambda_2 ; n_3 \cdot E \cdot n_3 = \lambda_3.\) It makes sense that if we only know the eigenvectors and want to undo this transformation to bring us back to \(E,\) then we need a transformation matrix that involves \(n_1, n_2, n_3.\) We can see that \(\Phi \cdot U \cdot \Phi^T\) is the opposite of the usual \(\Phi^T \cdot U \cdot \Phi,\) and without going through the rigorous derivation of why \(\Phi_{ii} = (n_1)_i, \Phi_{i2} = (n_2)_i, \Phi_{i3} = (n_3)_i,\) it at least makes some sense intuitively.

d) \([R]_e = [F]_e[U]^T_e =
\begin{bmatrix}
  .55 & .83 & 0 \\
  -.83 & .55 & 0 \\
  0 & 0 & 1
\end{bmatrix} \text{ (orthogonal ; }R^T = R^{-1})
\]

e) \([V]_e = [F]_e[R]^T_e =
\begin{bmatrix}
  3.05 & .83 & 0 \\
  .83 & .55 & 0 \\
  0 & 0 & 1
\end{bmatrix}
Chapter 3

Rates of Deformation

Typically, the equation of motion is solved in small increments of time, and the constitutive relationships are developed in rate form. “Rate” formulation in structural analysis is useful even if the actual “rates” involved are small or negligible, since incremental time stepping in time allows one to keep track of parameters that may be history dependent. This is the case, for example, in plasticity. At the other end of the spectrum, we may have a purely elastic problem, but one in which the rates are high enough that material parameters change (strain rate effects on stiffness). This cannot be considered unless we formulate our equations in rate form! These are merely examples that are meant to convey the importance of rates. Neither plasticity (metals), viscoelasticity (polymers), nor general rate effects (any material) will be considered in this text.

The rate form for hyperelastic materials, which forms the basis for the material modeling of rubber, require a special “hypoelastic” treatment. We will consider this in a later chapter. Furthermore, we will see that in linear infinitesimal elasticity, where the basic constitutive relationship is relatively simple, one has to still carefully consider factors such as change of reference frame due to rigid body rotations. Incremental or rate formulations handle such issues quite nicely.

The velocity at a point in the continuum, similar to the displacement at a point, depends on both position and time:
\[\mathbf{v} = \mathbf{v}(\mathbf{x}, t)\]
\[v_i = v_i(x_j, t)\]
3.1 Rate of Deformation and Spin Tensors

The most obvious first step is to take the time derivative of the deformation gradient, $F$, as follows:

$$\dot{F} = \frac{d}{dt} \frac{\partial x}{\partial X} = \frac{\partial}{\partial X} \left( \frac{dx}{dt} \right) = \frac{\partial}{\partial X} v$$

For reasons that won’t become clear until the chapter on rate-form constitutive expressions, we would like the “velocity gradient” to be expressed with respect to $x$ rather than $X$. Since $x$ is a function of $X$, we can use the chain rule to arrive at the following important expression:

$$\dot{F} = \frac{\partial}{\partial X} v = \frac{\partial v}{\partial x} \frac{\partial x}{\partial X}$$

This is quite academic (and potentially quite difficult) compared to the way that $L$ would be found in FEM, so such an example will not be given. However, $L$ is a very important quantity as we will see in the chapter on rate-form constitutive relationships.
3.1. RATE OF DEFORMATION AND SPIN TENSORS

The velocity gradient, $L$, is a very important quantity and can be decomposed as follows:

$$L = \frac{1}{2} (L + L^T) + \frac{1}{2} (L - L^T)$$  \hspace{1cm} (3.2)

note: Similar to eq. (3.2), in linear infinitesimal elasticity, we can split the displacement gradient into a strain tensor and rotation tensor that are symmetric and anti-symmetric, respectively.

note: Any tensor can be similarly split into its symmetric and anti-symmetric parts.

“Rate of Deformation Tensor” = symmetric $\rightarrow D^T = D$
“Spin Tensor” = anti-symmetric or “skew” $\rightarrow W^T = -W$

$W$ is a pure rigid body rotation. A complete proof of this can be found in [2]. It can also be easily shown that $W$ is, in fact, skew. This proof can be found in Appendix A.2.

Consider the time rate of change of length of a particular element:

$$\frac{d}{dt}(ds)^2 = \frac{d}{dt}(dx \cdot dx) = \frac{d}{dt}(dX \cdot F^T \cdot F \cdot dX) = dX \frac{d}{dt}(F^T \cdot F) \cdot dX$$
$$= dX \cdot \dot{C} \cdot dX$$

$$\frac{dC}{dt} = \dot{C} = (F^T \cdot F) = F^T \cdot F + F^T \cdot \dot{F} = F^T \cdot (F \cdot F^{-1} + F^{-T} \cdot \dot{F}^T) \cdot F$$
$$= F^T \cdot (L + L^T) \cdot F = 2 \cdot F^T \cdot D \cdot F$$

So, $\frac{d}{dt}(ds)^2 = 2dX \cdot F^T \cdot D \cdot F \cdot dX = 2dx \cdot D \cdot dx$

$$dx = ds\mathbf{n} \rightarrow \frac{d}{dt}(ds)^2 = 2(ds)\frac{d}{dt}(ds) = 2(ds)\dot{n} \cdot D \cdot n$$

$$\rightarrow \frac{d}{dt}(ds) = D_{nn}ds$$  \hspace{1cm} (3.3)

Rate of change of length only depends on $D$, not $W$.

If $D = 0 \rightarrow$ rigid body motions only.

note: We could have split up $F$ into a sum, but we wouldn’t have arrived at anything useful. The way that eq. (3.3) was derived only works due to the product rule of derivatives. Proving that $W$ represents “spin” is more difficult and won’t be shown here.
Similar to the shear strain, $E_{mn}$, we can again consider how the angle between two vectors, $\mathbf{m}$, and $\mathbf{n}$, changes under deformation (fig. 3.2).

$$\frac{d\phi}{dt} = 2D_{mn} \quad \text{(skipped work)} \quad (3.4)$$

Note: $\dot{\mathbf{E}} = \mathbf{F}^T \cdot \mathbf{D} \cdot \mathbf{F}$ (Recall also: $\mathbf{E} = \mathbf{F}^T \cdot \mathbf{e} \cdot \mathbf{F}$)

$\dot{\mathbf{E}}$ reduces to $\mathbf{D}$ for infinitesimal deformation (ignoring rigid body rotations), since $\mathbf{F} \rightarrow \mathbf{I}$.

In general, though, $\dot{\mathbf{E}} \neq \dot{\mathbf{e}} \neq \mathbf{D}$. Similarly, in general, $\dot{\mathbf{R}} \neq \mathbf{W}$. We’ll also look in detail at the physical meaning (or lack thereof) of $\dot{\mathbf{\sigma}}$ later on when we get to the chapter on “Rate-Form Constitutive Expressions”

### 3.2 Other Rates of Change (Volume and Area)

For completeness, we’ll quickly derive the rate of volume change and the rate of area change:

**Volume:**

If $dV = ds_1ds_2ds_3$, where $ds_1$, $ds_2$, $ds_3$ are the lengths of the sides of a “box” that is oriented in the principal directions of $\mathbf{D}$, then:

$$\frac{d}{dt}(dV) = \frac{d(ds_1)}{dt}ds_2ds_3 + \frac{d(ds_2)}{dt}ds_1ds_3 + \frac{d(ds_3)}{dt}ds_1ds_2$$

$$= (D_{11} + D_{22} + D_{33})ds_1ds_2ds_3 = D_{kk}dV \quad \text{note:} \text{tr}(\mathbf{D}) = D_{kk} \quad (3.5)$$

**Area:**

Consider the following time derivative:

$$\frac{d}{dt}(dS\mathbf{n}) = \frac{dS}{dt}\mathbf{n} + dS\frac{\mathbf{d}\mathbf{n}}{dt}$$
where \( dS \) is an area (as opposed to \( ds_1, ds_2, ds_3 \), which are lengths).

\( dS \) is known from eq. (2.10). So, we can take the time derivative. We can also re-write \( \frac{d\mathbf{n}}{dt} \) in terms of \( D_{nn} \).

\[
\rightarrow \frac{d}{dt}(dS) = (D_{kk} - \underbrace{D_{nn}}_{\text{no sum}}) dS \quad \text{(skipped work)} \tag{3.6}
\]

We’ll come back to rates again when we get to rate-form constitutive relationships.
Chapter 4

Stress

Stress, like strain, has intuitive physical meaning for engineers. Stress can come in the form of tension, compression, or shear. The forces present on one or more elements within a body are found from stresses by integrating over areas, and quantities such as moments, though not found naturally in FEM, can be easily back-calculated from stresses as well.

Note: While “moments” are essential in classical structural analysis and design, from an FEM point-of-view (“solid” elements), they are an unnecessary simplification and will not be discussed in this text.

Forces (or stresses) within a body are found from externally applied forces, even if the objective is to find deformations. The stiffness of the material, which relates internal force and internal deformation, determines any deformations of interest that result from externally applied forces and body forces. This may all be familiar, since classical, “analytical,” structural analysis methods also require stiffness relationships that relate internal forces to displacements or moments to rotations. In FEM, stress and strain are used, directly, for this purpose, and the “stiffness” relationship that relates them is called a constitutive relationship. This will be discussed in great detail beginning in the next chapter. This chapter will just focus on stress. Specifically, we’ll consider how to physically interpret a stress tensor, along with some different measures of stress that are commonly used, and how they each handle considerations like large deformations and rigid body rotations.

The “Cauchy” stress is a second-order, symmetric, tensor that contains six...
CHAPTER 4. STRESS

independent components: 3 axial stresses and 3 shear stresses. More on the physical description of $\sigma$ can be found in Appendix B.1, where $\sigma$ is essentially derived.

Simple 2D e.x.
Imagine that at some point within some structure, we’ve determined the stresses to be (ignore units):

$$
\begin{align*}
\sigma_{11} &= 12,300 & \sigma_{22} &= -4,200 & \sigma_{33} &= 0 \\
\sigma_{32} &= \sigma_{23} = 0 & \sigma_{31} &= \sigma_{13} = 0 & \sigma_{21} &= \sigma_{12} = -4,700 
\end{align*}
$$

To find the stresses, at, say, 45°, we could use Mohr’s Circle or the transformation equations from undergraduate “Mechanics of Materials.”

Solution for the normal stress at 45° for example:

$$
\sigma_{xx} = 650 \text{ (skipped work)}
$$

Using the stress tensor method instead:

$$
n = [.707, .707, 0]$$

$$
\sigma \cdot n = 
\begin{bmatrix}
.707(\sigma_{11}) - .707(\sigma_{12}) + 0 \\
.707(\sigma_{21}) - .707(\sigma_{22}) + 0 \\
.707(0) - .707(0) + 0
\end{bmatrix}
$$

$$
n \cdot \sigma \cdot n = \sigma_{nn} = “\sigma_{xx} \text{ at } 45°” = 650, \text{ as expected}
$$

4.1 Cauchy Equation of Motion

The equation of motion can be expressed in terms of the applied stress, body forces, mass, and acceleration:

$$
\int_S t dS + \int_V \rho b dV = \frac{d}{dt} \int_V \rho v dV ; \ t = \sigma \cdot n
$$

(4.1)

In index notation: $\int_S t_i dS + \int_V \rho b_i dV = \frac{d}{dt} \int_V \rho v_i dV ; \ t_i = \sigma_{ij} n_j$
4.1. CAUCHY EQUATION OF MOTION

or

\[ \int_S \sigma_{ij} n_j dS + \int_V \rho b_i dV = \int_V \rho \frac{dv_i}{dt} dV \quad (4.2) \]

Note that the step from eq. (4.1) to eq. (4.2) is not trivial, since both \( \rho \) and \( dV \) do in fact change with time. The complete derivation, starting with eq. (4.1) and concluding with eq. (4.2), is given in Appendix B.2.

Thus,

\[ \int_V \left( \frac{\partial \sigma_{ij}}{\partial x_j} + \rho b_i - \rho \frac{dv_i}{dt} \right) dV = 0 \quad (4.3) \]

Note that the “localization theorem” states that \( \int_a^b f dx = 0 \rightarrow f = 0 \), if \( a \) and \( b \) are arbitrary.

\[ \frac{\partial \sigma_{ij}}{\partial x_j} + \rho b_i = \rho \frac{dv_i}{dt} \quad (4.4) \]

Eq. (4.4) are the Cauchy Equations of Equilibrium. It may not yet be clear which term contains the applied forces and which term contains the quantities analogous to “\( k \star x \).” It turns out that the \( \sigma_{ij} \) term will contain applied forces and prescribed displacements, along with all of the internal force and displacement quantities that constitute “\( k \star x \)” for the element. When an entire system is analyzed, which includes many elements, the global equation of motion should be satisfied, naturally, so long as the geometry is accurately represented by the elements, the stiffness and strength properties of the material are defined for each element, and the boundary conditions are correctly assigned for each element. There could be other issues that arise as well, due to simplifications inherent (but quite necessary) in the finite element method (FEM), but these issues will be left to texts that cover FEM in detail. In fact, among the aforementioned element-related issues, this text will only cover material elastic stiffness in detail. Material behavior at the limit state (failure) is covered in texts on plasticity, for example, and topics relating to element geometries, prescribed degrees of freedom or prescribed forces at “nodes” (i.e. boundary conditions), or other issues related to the “assembly” of finite elements will be left to texts devoted to the topic of
FEM implementation.

If \( v_i = 0 \), then eq. (4.4) reduces to static equilibrium.

\[
\left( \text{static equilibrium: } \frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{12}}{\partial x_2} + \frac{\partial \sigma_{13}}{\partial x_3} = 0 \right) \quad \text{(no sum on } i) \]

note: This is the basic differential equation used in FEM, though the connection to FEM will not really be clear until we start developing constitutive equations relating stresses and strains (along with the above equation and strain-displacement relationships previously presented).

### 4.2 Alternative Measures of Stress

Stress can be defined in the “spatial” coordinate system or in the “material” coordinate system (note that these two coordinate systems differ when rigid body rotation is present). In addition, stress can be defined as force per unit deformed area or force per unit initial area.

\[
\text{df}_n = t_n \cdot dS
\]

where \( t_n = n \cdot \sigma \)

Now, define \( \text{df}_n = t^0_n dS_0 \), where \( t^0_n = N \cdot \sigma \)

(the actual force, on the undeformed area, gives us the “nominal” stress \( \sigma^0 \))

\[
\text{df}_n = t^0 dS_0 = t dS
\]

\[
dS_0 N \cdot \sigma^0 = dS n \cdot \sigma
\]
4.2. ALTERNATIVE MEASURES OF STRESS

Recall Nanson’s Equation (eq. (2.10)): 
\[ dS = (\text{det}F)N \cdot F^{-1}dS_0 \]

\[ dS_0 \cdot \sigma^0 = (\text{det}F)dS_0 \cdot F^{-1} \cdot \sigma \]

(more formally: \[ dS_0 \cdot [\sigma^0 - (\text{det}F)F^{-1} \cdot \sigma] = 0 \])

\[ \sigma^0 = (\text{det}F)F^{-1} \cdot \sigma \] \hspace{1cm} (4.5)

Eq. (4.5) is the Nominal Stress Tensor

Now, consider a “pseudo-force” acting in the initial frame of reference (Fig. 4.2).

![Figure 4.2: 2nd Piola-Kirchhoff stress](image)

note: The “pseudo-force” pictured in Fig. 4.2 will become more clear later.
The following derivation of the “Second Piola-Kirchhoff” (P-K II) stress tensor is similar to the previous derivation of the “Nominal” stress tensor. The P-K II tensor will be quite useful later on to help us derive constitutive relationships.

Recall \( dx = F \cdot dX \)

Similarly, \( df_n = F \cdot df_n \); \( \hat{d}f_n = \hat{t}_n dS_0 \); \( \hat{t}_n = N \cdot \hat{\sigma} ; \hat{\sigma} = \text{P-K II tensor} \)

We already know that \( df_n = t_n \cdot dS = n \cdot \sigma dS \)

So, \( df_n = F \cdot \hat{df}_n = F \cdot \hat{t}_n dS_0 = \hat{t}_n \cdot F^T dS_0 = N \cdot \hat{\sigma} \cdot F^T dS_0 \)

Substituting for \( df_n \), \( n \cdot \sigma dS = N \cdot \hat{\sigma} \cdot F^T dS_0 \) \hspace{1cm} (1)

Invoke Nanson’s equation: \( ndS = (\text{det}F)N \cdot F^{-1}dS_0 \) \hspace{1cm} (2)
(2) $\rightarrow$ (1) $\rightarrow$ $(\text{det} F) N d S_0 \cdot F^{-1} \cdot \sigma = N d S_0 \cdot \hat{\sigma} \cdot F^T$
(more formally, $N d S_0 \cdot [(\text{det} F) F^{-1} \cdot \sigma - \hat{\sigma} \cdot F^T] = 0$)

$$\rightarrow \hat{\sigma} = (\text{det} F) F^{-1} \cdot \sigma \cdot F^{-T}$$ (4.6)

Eq. (4.6) is the 2\textsuperscript{nd} Piola-Kirchhoff Stress Tensor

For infinitesimal deformation (ignoring rigid body rotations):
$F \approx I$ ; $\text{det} F = 1$ ; $\sigma \approx \sigma^0 \approx \hat{\sigma}$

\textit{e.x.} The deformed equilibrium configuration of a body is defined by the deformation mapping:

$x_1 = -\frac{1}{2} X_1$, $x_2 = \frac{1}{2} X_3$, $x_3 = 2 X_2$

(Very large deformations and very large rigid body motions)

![Figure 4.3: Illustration of deformation](image)

The Lagrangian Strain is:

$$E = \frac{1}{2} (F^T F - I) = \begin{bmatrix}
-0.375 & 0 & 0 \\
0 & 1.5 & 0 \\
0 & 0 & -0.375
\end{bmatrix}$$

$$e = \frac{1}{2} (I - F^{-T} F^{-1}) = \begin{bmatrix}
-1.5 & 0 & 0 \\
0 & -1.5 & 0 \\
0 & 0 & 0.375
\end{bmatrix}$$
4.2. ALTERNATIVE MEASURES OF STRESS

Note that $E_{22} = 1.5$ (this is the “$2$” direction; initially the “$y$” direction). The specimen did not elongate from left to right, yet $E_{22}$ is a positive value. In addition, $E_{11}$ and $E_{33}$ are the same, confirming that they represent the transverse strains. The axes actually underwent a large rigid body rotation. $E$ is a strain measure that rotates with the axes. In other words, $E$ is invariant to rigid body rotations. This will be essential to remember in the later chapters!

The Cauchy Stress is:

$$\sigma = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 100 & 0 \end{bmatrix} \text{ (ignore units)}$$

Note that $\sigma$ is defined in a different coordinate system than $E$ (“spatial” axes rather than “material” axes). $\sigma$ changes with rigid body rotation, while $E$ does not!

a) Determine the nominal and P-K II stress tensors
b) The nominal and P-K II tractions associated with the plane $x_3=\text{const}$ in the undeformed state
c) Repeat parts “a” and “b” for the mapping:
   $x_1 = -.99X_1$; $x_2 = .99X_3$; $x_3 = .99X_2$ (small deformations with very large rigid body motions)

a) and b):

Since the deformations are given, we first find $F$:

$$F = \begin{bmatrix} \frac{\partial x_1}{\partial X_1} & \frac{\partial x_1}{\partial X_2} & \frac{\partial x_1}{\partial X_3} \\ \frac{\partial x_2}{\partial X_1} & \frac{\partial x_2}{\partial X_2} & \frac{\partial x_2}{\partial X_3} \\ \frac{\partial x_3}{\partial X_1} & \frac{\partial x_3}{\partial X_2} & \frac{\partial x_3}{\partial X_3} \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} \\ 0 & 2 & 0 \end{bmatrix}$$

With $F$ and $\sigma$ known, we can find $\sigma^0$

$$\sigma^0 = (\det F) F^{-1} \cdot \sigma = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 25 \\ 0 & 0 & 0 \end{bmatrix}$$

Remember, this stress is a result of the actual force on the undeformed area.

From Nanson’s Equation (eq. (2.10)),

$$\sigma^0 = (\det F) F^{-1} \cdot \sigma = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 25 \\ 0 & 0 & 0 \end{bmatrix}$$
CHAPTER 4. STRESS

\[ dA_0 \mathbf{N} = \frac{1}{(\det \mathbf{F})} \mathbf{F}^T \cdot \mathbf{n} = \frac{1}{2} \mathbf{F}^T \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = 4 \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \]

where \( \begin{bmatrix} 0 \\ 1 \end{bmatrix} \) is the normal to the plane \( x_3=\text{constant} \)

Since we know from inspection that \( \mathbf{N} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \), we can see that the undeformed area is four times that of the deformed area, hence we expect the nominal stress to be four times smaller that the true (Cauchy) stress, which it is.

We have everything we need in order to find \( \hat{\mathbf{\sigma}} \):

\[
\hat{\mathbf{\sigma}} = (\det \mathbf{F}) \mathbf{F}^{-1} \cdot \mathbf{\sigma} \cdot \mathbf{F}^{-T} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 12.5 & 0 \\ 0 & 0 & 0 \end{bmatrix}
\]

Traction on the plane \( x_3=\text{constant} \):

\[
\hat{\mathbf{t}} = \hat{\mathbf{\sigma}} \cdot \mathbf{N} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 12.5 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -12.5 \\ 0 \end{bmatrix}
\]

c) What about small deformation (still large rigid body rotation)?

\[
\mathbf{F} = \begin{bmatrix} -0.99 & 0 & 0 \\ 0 & 0 & 0.99 \\ 0 & 1.01 & 0 \end{bmatrix}
\]

\[
\mathbf{E} = \begin{bmatrix} -0.00995 & 0 & 0 \\ 0 & 0.01005 & 0 \\ 0 & 0 & -0.00995 \end{bmatrix}
\]

\[
\mathbf{e} = \begin{bmatrix} -0.01015 & 0 & 0 \\ 0 & -0.01015 & 0 \\ 0 & 0 & 0.00985 \end{bmatrix}
\]

\[
\mathbf{\sigma}^0 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 98 & 0 \\ 0 & 0 & 0 \end{bmatrix}; \hat{\mathbf{\sigma}} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 97 & 0 \\ 0 & 0 & 0 \end{bmatrix}
\]
4.3. PRINCIPAL STRESSES

Note that \( \hat{\sigma} \) gives a good answer for small deformations, and is defined in the same coordinate system as \( \mathbf{E} \!\).

4.3 Principal Stresses

Again, let’s start with \( t_n \):

\[
\mathbf{t}_n = \mathbf{n} \cdot \mathbf{\sigma}
\]

\[
\sigma_{nn} = \mathbf{t}_n \cdot \mathbf{n} = \mathbf{n} \cdot \mathbf{\sigma} \cdot \mathbf{n}
\]

For what \( \mathbf{n} \) is \( \sigma_{nn} \) maximized?

Remember, \( \mathbf{n} \cdot \mathbf{n} = 1 \)

“Lagrange multiplier” = \( \phi = \sigma_{nn} - \lambda (\mathbf{n} \cdot \mathbf{n} - 1) \)

In index notation, \( \phi = n_i \sigma_{ij} n_j - \lambda (n_i n_i - 1) \)

\[
\frac{\partial \phi}{\partial n_k} = 0 \text{ to find local optimums}
\]

\[
\begin{align*}
\frac{\partial n_i}{\partial n_k} (\sigma_{ij} n_j) + (n_i \sigma_{ij}) & \frac{\partial n_j}{\partial n_k} - \lambda \frac{\partial n_i}{\partial n_k} n_i - \lambda \frac{\partial n_i}{\partial n_k} n_i = 0 \\
\end{align*}
\]

Simplifying, we get:

\[
\begin{align*}
\sigma_{kj} n_j + n_i \sigma_{ik} - 2\lambda n_k &= 0 \quad \rightarrow 2\sigma_{ik} n_i - 2\lambda n_k = 0 \\
\end{align*}
\]

This further reduces to:

\[
\begin{align*}
\sigma_{ik} n_i - \lambda n_k &= 0 \quad \rightarrow t_k = \lambda n_k \\
\end{align*}
\]

(\( \mathbf{\sigma} - \lambda \mathbf{I} \) \cdot \mathbf{n} = 0 \rightarrow Eigenvalue problem \rightarrow \lambda \) is precisely \( \sigma_{\text{max}, \text{min}} \)

And, at the principal plane, traction is in the direction of the normal -i.e. no shear. (max shear = \( \frac{\sigma_{\text{max}} - \sigma_{\text{min}}}{2} \))

Simple 2D e.x.: Problems like the following are typical in undergrad “mechanics of materials”, where Mohr’s Circle would be used to find the maximum stresses. Tensor methods are faster and can be more easily extended to 3D.
\[ \sigma = \begin{bmatrix} -4200 & -4700 & 0 \\ -4700 & 12300 & 0 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \lambda_1 = -5445; \; \lambda_2 = 13540 \]

Principal direction, \( n_{\lambda_1} = \begin{bmatrix} -0.97 \\ -0.26 \\ 0 \end{bmatrix} \)

\[ \theta_p = tan^{-1} \frac{0.26}{0.97} = 75^\circ \]

The above values of stress and angle agree with the transformation equations from undergrad, of course.
Chapter 5

Superimposed Rotation

In preparation for the constitutive formulations in later chapters, this short chapter will derive how different tensors behave under superimposed rigid body motion. This will enable us, for example, to show that $\sigma$ is a function of $B$ in the beginning of the next chapter. The idea of “work conjugate” pairs will be introduced in this chapter. We will see that the Cauchy stress tensor, $\sigma$, for example, changes with rigid body rotation, while $E$ does not. Thus, $\sigma$ and $E$ are not work-conjugate, and therefore it would not be appropriate to use them as a pair when forming a constitutive relationship.

Let’s define the vector $dx^*$ as follows:

\[ dx^* = Q \cdot dx, \text{ where } Q \text{ is a superimposed rigid body motion (Fig. 5.1).} \]

We know that $dx = F \cdot dX$.

What about “$F^*$”? Consider the following square body (Fig. 5.2), paying careful attention to the heavily bolded edge in order to track the motion of the body.

Clearly, if $dX = dX^*$ (which is what we’re going to want), and, say, $F^* = F$, then $dx^* \neq F^* \cdot dX^*$ (this should be clear from Fig. 5.2).

Since we want to find $F^*$ such that $dx^* = F^* \cdot dX^*$, we can conclude that $F^* \neq F$.

What we do know is the following:
CHAPTER 5. SUPERIMPOSED ROTATION

Figure 5.1: Superimposed rigid body motion

Figure 5.2: Superimposed rigid body motion

"F" only stretches in this case, for the purposes of illustration.
\[ dX^* = dX \; ; \; dx^* = Q \cdot dx \; ; \; dx^* = F^* \cdot dX^* \]

So, \( Q \cdot dx = F^* \cdot dX \), which gives:

\[ F^* = Q \cdot F \tag{5.1} \]

Eq. (5.1) is what we wanted to find, and it should be expected. Recall that \( R \cdot \textbf{U} \) is physically understood to be a deformation (axial strains and shear), \( \textbf{U} \), followed by a rotation, \( R \). Thus, \( Q \cdot F \) is the total deformation + rotation, \( F \), followed by an explicit rigid body rotation, \( Q \), - i.e. a rotation that is superimposed on \( dx \).

Now, we can see how different measures of strain and stress behave under superimposed rigid body motion, by substituting the above expression for \( F^* \).

\[ C = F^T \cdot F \longrightarrow C^* = F^{*T} \cdot F^* = F^T \cdot Q^T \cdot Q \cdot F = F^T \cdot F = C \tag{5.2} \]

Thus, \( C \) is invariant to rigid body motion.

\[ \text{note: } A^{-1} \cdot A = I \text{ always } ; \; Q^T \cdot Q = I \text{ since } Q \text{ is an orthogonal tensor} \]
CHAPTER 5. SUPERIMPOSED ROTATION

Note: $F^T = (Q \cdot F)^T = F^T \cdot Q^T$

Informal proof: $B = B_{ik}e_ie_k; C = C_{nj}e_ne_j$

$A = B \cdot C = B_{ik}e_ie_k \cdot C_{nj}e_ne_j = B_{ik}C_{kj}e_ie_j = A_{ij}e_ie_j$

$A^T = A_{ij}e_ie_j = B_{ik}C_{kj}e_ie_j$

or

$A^T = A_{ji}e_ie_j = B_{jk}C_{ki}e_ie_j$


$A^T$ NOT $= B^T \cdot C^T = B_{ki}e_ie_k \cdot C_{jn}e_ne_j = B_{ki}C_{jk}e_ie_j$ which does not match either of the above expressions for $A^T$

(or $B^T \cdot C^T = B_{ik}e_ik \cdot C_{nj}e_ne_j = B_{ik}C_{ni}e_ne_n = B_{ki}C_{jk}e_ie_j$ which is similarly $\neq A^T$)

$A^T = C^T \cdot B^T = C_{jn}e_ne_j \cdot B_{ki}e_ik = C_{kn}B_{ki}e_ie_i \rightarrow$ matches 1st expression for $A^T$

or $A^T = C^T \cdot B^T = C_{nj}e_je_n \cdot B_{ki}e_ik = C_{kj}B_{ik}e_ie_i = C_{ki}B_{jk}e_ie_j \rightarrow$ matches 2nd expression for $A^T$


Formal proof: see Appendix A.1

\[ B = F \cdot F^T \longrightarrow B^* = F^* \cdot F^{*T} = Q \cdot F \cdot F^T \cdot Q^T = Q \cdot B \cdot Q^T \quad (5.3) \]

What about $U^*$, $V^*$, and $R^*$?

$F = V \cdot R = R \cdot U \longrightarrow F^* = V^* \cdot R^* = R^* \cdot U^*$

Recall, $F^* = Q \cdot F = Q \cdot V \cdot R$

Insert $Q^T \cdot Q = I$ into the above expression $\longrightarrow F^* = \underbrace{Q \cdot V \cdot Q^T}_{V^*} \cdot \underbrace{Q \cdot R}_{R^*}$

So, $V^* = Q \cdot V \cdot Q^T ; R^* = Q \cdot R$ ($V^*$ is symmetric; $R^*$ is orthogonal)

Also, $U^* = U$ (skipped work)
How about rates?

\[ \mathbf{F}^* = \mathbf{Q} \cdot \mathbf{F} \quad ; \quad \dot{\mathbf{F}}^* = \dot{\mathbf{Q}} \cdot \mathbf{F} + \mathbf{Q} \cdot \dot{\mathbf{F}} \]

Recall also, \( \mathbf{L} = \dot{\mathbf{F}} \cdot \mathbf{F}^{-1} \) ; \( \mathbf{L}^* = \dot{\mathbf{F}}^* \cdot \mathbf{F}^{*-1} \) ; \( \mathbf{F}^{*-1} = \mathbf{F}^{-1} \cdot \mathbf{Q}^{-1} \)

So, \( \mathbf{L}^* = (\dot{\mathbf{Q}} \cdot \mathbf{F} + \mathbf{Q} \cdot \dot{\mathbf{F}}) \cdot \mathbf{F}^{-1} \cdot \mathbf{Q}^{-1} = \dot{\mathbf{Q}} \cdot \mathbf{Q}^{-1} + \mathbf{Q} \cdot \left( \frac{\dot{\mathbf{F}}}{\mathbf{L}} \cdot \mathbf{F}^{-1} \right) \cdot \mathbf{Q}^T \)

\[ \text{note: } (\mathbf{Q}^{-1} = \mathbf{Q}^T \text{ since } \mathbf{Q} \text{ is orthogonal}) \]

\[ \mathbf{L}^* = \dot{\mathbf{Q}} \cdot \mathbf{Q}^{-1} + \mathbf{Q} \cdot (\mathbf{D} + \mathbf{W}) \cdot \mathbf{Q}^T = \underbrace{\mathbf{Q} \cdot \mathbf{D} \cdot \mathbf{Q}^T}_{\mathbf{D}^*} + \underbrace{\dot{\mathbf{Q}} \cdot \mathbf{Q}^{-1} + \mathbf{Q} \cdot \mathbf{W} \cdot \mathbf{Q}^T}_{\mathbf{W}^*} (5.4) \]

\[ \text{note: } \mathbf{W}^* \text{ is antisymmetric, and indeed, the two terms that make up } \mathbf{W}^* \]

are each antisymmetric

How about stress?

Since \( d \mathbf{f}_n^* = \mathbf{Q} \cdot d \mathbf{f}_n \), we can say that \( \mathbf{t}_n^* = \mathbf{Q} \cdot \mathbf{t}_n \quad (1) \)

We also know the following three expressions to be true:

\[ \mathbf{t}_n = \mathbf{\sigma} \cdot \mathbf{n} \quad (2) \]
\[ \mathbf{t}_n^* = \mathbf{\sigma}^* \cdot \mathbf{n}^* \quad (3) \]
\[ \mathbf{n}^* = \mathbf{Q} \cdot \mathbf{n} \quad (4) \]

(4) \( \rightarrow \) (1) \( \rightarrow \) (3) \( \rightarrow \) \( \mathbf{Q} \cdot \mathbf{t}_n = \mathbf{\sigma}^* \cdot \mathbf{Q} \cdot \mathbf{n} \quad (5) \)

(2) \( \rightarrow \) (5) \( \rightarrow \) \( (\mathbf{Q} \cdot \mathbf{\sigma}) \cdot \mathbf{n} = (\mathbf{\sigma}^* \cdot \mathbf{Q}) \cdot \mathbf{n} \)

\[ \mathbf{Q} \cdot \mathbf{\sigma} = \mathbf{\sigma}^* \cdot \mathbf{Q} \rightarrow \mathbf{\sigma}^* = \mathbf{Q} \cdot \mathbf{\sigma} \cdot \mathbf{Q}^T \quad (5.5) \]

\[ \text{note: Since } \mathbf{\sigma}^* = \mathbf{Q} \cdot \mathbf{\sigma} \cdot \mathbf{Q}^T \text{ and } \mathbf{B}^* = \mathbf{Q} \cdot \mathbf{B} \cdot \mathbf{Q}^T, \text{ we can say that } \mathbf{\sigma} \text{ and } \mathbf{B} \]

are a “work-conjugate pair.”

How about the nominal stress and the Second Piola-Kirchhoff Stress?

\[ \mathbf{\sigma}^0 = (\text{det} \mathbf{F}) \cdot \mathbf{F}^{-1} \cdot \mathbf{\sigma} \]
Again, we will need to use \( F^* = Q \cdot F \)

Since \( \sigma^0 \) depends on \( \det F \), we will need \( \det F^* \). It turns out, though, that
\[
\det F^* = (\det Q)(\det F) = \det F
\]

\[ \text{note: the determinant of an orthogonal tensor is “1”} \]

So,
\[
\sigma^{0*} = (\det F^*)F^{*-1} \cdot \sigma^* = \det F \cdot (Q \cdot F)^{-1} \cdot Q \cdot \sigma \cdot Q^T
\]
\[
= (\det F)F^{-1} \cdot Q^{-1} \cdot Q \cdot \sigma \cdot Q^T = (\det F)F^{-1} \sigma \cdot Q^T = \sigma^0 \cdot Q^T
\]

How about \( \hat{\sigma} \)?

\[
\hat{\sigma} = \sigma^0 \cdot F^{-T}
\]
\[
\hat{\sigma}^* = \sigma^{0*} \cdot (F^*)^{-T} = \sigma^0 \cdot Q^T \cdot (Q \cdot F)^{-T} = \sigma^0 \cdot Q^T \cdot Q \cdot F^{-T} = \sigma^0 \cdot F^{-T} = \hat{\sigma}
\]

\[
\hat{\sigma}^* = \hat{\sigma}
\]

\( \hat{\sigma} \) is a good measure of stress when there are small (infinitesimal) strains. It will also be used as a “work-conjugate pair” with \( E \) in the following chapters on constitutive relationships.
Chapter 6

Hyperelasticity

We’ve derived various measures of strain based on deformations that are presumably known. We’ve also derived various measures of stress, emphasizing the physical meaning of the stress tensor, for the case when normal and shear stresses on a body are known. To solve problems where the objective is to find deformations (strains) and internal stresses within a structure, knowing, for example, the forces externally applied to the structure, we need to know the stiffnesses of the materials involved. Just as the stiffness of a spring relates force to deformation, as Hooke famously showed in 1678 “ut tensio, sic vis,” meaning, “as the extension, so the force,” the stiffness of a material can be thought of as relating stress to strain. This relationship is called a constitutive relationship.

While many engineering materials, in the elastic regime, can be idealized as linear and only undergo infinitesimal strain, many other types of materials, like rubbery elastomers, behave nonlinearly and reversible (elastic) even when subjected to very large deformations. While in practice, such materials typically display viscoelastic behavior even at low strain rates, at infinitesimally slow strain rates, they approach hyperelastic behavior. Moreover, hyperelastic material models underlie even the most complex models that incorporate viscoelasticity, damage, etc. Thus, hyperelasticity is essential for modeling polymers in FEM. Several specific kinds of rubber models will be considered in examples at the end of this chapter.

A hyperelastic model is a general constitutive relationship, relating stress and strain, for large deformation (large strain), nonlinear, elastic (reversible), isotropic, materials. While different materials can be best characterized by
different “strain energy density functions,” and certainly have different material constants (found via experimentation), our general constitutive expression will be derived in closed form, and in 3D.

It may be useful to note that “curve-fitting” techniques that finds engineering stress as a function of engineering strain, based on uniaxial stress-strain experimental data, for example, are becoming more common. Extending such a method to accommodate deformation in three dimensions is a challenge that usually necessitates that we start with a 3D “hyperelastic” energy function. As we will see in the examples at the end of this chapter, “hyperelastic” functions that relate stress and strain, while nonlinear in nature, actually do not require much higher order algebra, yet have been proven to work for a wide variety of hyperelastic materials. We will see that these functions are fairly simple in the sense that they require the determination of only a few experimental constants. While these experimental constants are typically found from uniaxial tests or shear tests, the hyperelastic constitutive relationships are always written in tensor form (3D), using stress and strain tensors that form a work-conjugate pair. This enables our hyperelastic constitutive relationships to be implemented into FEM quite naturally.

Hyperelastic functions fall into two major categories: micro-mechanical models and phenomenological models. This will be discussed toward the end of this chapter, and several hyperelastic functions will be presented. In addition, it is important to be able to fully characterize a particular material once a hyperelastic function has been chosen. Experimental data is necessary for this purpose, and several methods for efficiently characterizing materials will be demonstrated. Two methods, which are particularly “cutting edge,” will be presented at the conclusion of this chapter.

This may be an appropriate time to point out that this text will develop a constitutive relationship for two branches of elasticity: hyperelasticity (this chapter) and linear infinitesimal elasticity (final chapter). Given that hyperelasticity encompasses elastic material nonlinearity under large strain, one might ask how we should approach a material that is elastic linear under large strain, or elastic nonlinear under small strain. It turns out that these special cases would be treated just like any nonlinear hyperelastic material - i.e. one would use the basic hyperelastic constitutive relationship that will be derived in this chapter. The constitutive relationship that will be derived in the final chapter, on linear infinitesimal elasticity, will be valid only for materials that are linear and subjected to small deformation.
The following will simply be stated, but a proof can be found in Appendix C.1:

\[ \sigma = g(B) \]  \hspace{1cm} (6.1)

|note: Eq. (6.1) is perhaps obvious since \( \sigma^* = Q \cdot \sigma \cdot Q^T \) and \( B^* = Q \cdot B \cdot Q^T \) from last chapter (we called this a work-conjugate pair based on derivation from Q superimposed on \( dx \)). |

Now, we know that the strain energy is related to the product of force and deformation.

So, let’s define \( \phi \) to be the strain energy (per unit volume)

\[ \hat{\sigma} = \frac{d\phi}{dE} = 2 \frac{d\phi}{dC}, \text{ since } E = \frac{1}{2}(C - I) \]

Remember, \( \hat{\sigma} \) and \( E \) (or \( C \)) are a work-conjugate pair and from eq. (4.6) we know:

\[ \hat{\sigma} = (\det F)F^{-1} \cdot \sigma \cdot F^{-T} = 2 \frac{d\phi}{dC} \]
\[ \longrightarrow \sigma = 2 \frac{d\phi}{dC} \cdot F \cdot F^T \]

\[ \frac{d\phi}{dC} = R^T \cdot \frac{d\phi}{dB} \cdot R ; \text{ The proof is straightforward (can be inferred from Appendix C.1), so long as it is recognized that } \phi = \phi(\xi), \text{ where } \xi \text{ can be replaced by either } C \text{ or } B, \text{ as they have the same eigenvalues and invariants, for example.} \]

So, \[ 2 \frac{d\phi}{dC} \cdot F \cdot F^T = 2 \frac{d\phi}{dC} \cdot V \cdot \frac{R \cdot R^T \cdot \frac{d\phi}{dB} \cdot R \cdot R^T}{I} \cdot V \]
\[ \det F = \det(V \cdot R) = \det V, \text{ since } \det R = 1 \text{ (rigid body rotation)} \]
\[ (\det V = \det F = 1 \text{ if material is incompressible}) \]

So,

\[ \sigma = \frac{2}{\det V} V \cdot \frac{d\phi}{dB} \cdot V \]  \hspace{1cm} (6.2)

Since \( B = V^2 \longrightarrow \sigma = \frac{2}{\det(B^2)} B \cdot \frac{d\phi}{dB} \)
or
(symmetrized form)

\[
\sigma = \frac{1}{\det B^{1/2}} \left( B \frac{d\phi}{dB} + \frac{d\phi}{d\mathbf{B}} \mathbf{B} \right) \tag{6.3}
\]

Eq. (6.3) is possible since \( B \frac{d\phi}{dB} = \frac{d\phi}{d\mathbf{B}} \mathbf{B} \)

Now, we know that for isotropy it is possible to write \( \phi = \phi(I_B, II_B, III_B) \), where \( I_B, II_B, III_B \) are each a function of \( \mathbf{B} \) [30]. Since we will look at specific strain energy functions, \( \phi \), later on, which happen to be of such a form, we need to apply the chain rule to \( \frac{d\phi}{d\mathbf{B}} \).

\[
\rightarrow \frac{d\phi}{d\mathbf{B}} = \frac{\partial \phi}{\partial I_B} \frac{dI_B}{d\mathbf{B}} + \frac{\partial \phi}{\partial II_B} \frac{dII_B}{d\mathbf{B}} + \frac{\partial \phi}{\partial III_B} \frac{dIII_B}{d\mathbf{B}} \tag{6.4}
\]

We need to determine \( \frac{dI_B}{d\mathbf{B}}, \frac{dII_B}{d\mathbf{B}}, \) and \( \frac{dIII_B}{d\mathbf{B}} \):

The general expressions for the second order tensor invariants \( I_B \) and \( II_B \) will just be stated, as it is a math topic:

\[
I_B = \text{tr } \mathbf{B} \quad ; \quad II_B = \frac{1}{2}[(\text{tr } \mathbf{B})^2 - \text{tr}(\mathbf{B}^2)] \quad \text{(skipped work)}
\]

The complete derivations of \( \frac{dI_B}{d\mathbf{B}}, \frac{dII_B}{d\mathbf{B}}, \) and \( \frac{dIII_B}{d\mathbf{B}} \) are given in Appendix C.2.

\[
\frac{dI_B}{d\mathbf{B}} = \mathbf{I} \tag{6.5}
\]

\[
\frac{dII_B}{d\mathbf{B}} = I_B \mathbf{I} - \mathbf{B} \tag{6.6}
\]

\[
\frac{dIII_B}{d\mathbf{B}} = \mathbf{B}^2 - I_B \mathbf{B} + II_B \mathbf{I} \tag{6.7}
\]
\[ \frac{d\phi}{dB} = \frac{\partial \phi}{\partial I_B} I_B + \frac{\partial \phi}{\partial II_B} (I_B I - B) + \frac{\partial \phi}{\partial III_B} (B^2 - I_B B + II_B I) \quad (6.8) \]

Recall \( \sigma = \frac{1}{\det B^{1/2}} (B \frac{d\phi}{dB} + \frac{d\phi}{dB} B) \)

Substituting:
\[
\frac{1}{\det B^{1/2}} \left( B \left[ \frac{\partial \phi}{\partial I_B} I_B + \frac{\partial \phi}{\partial II_B} (I_B I - B) + \frac{\partial \phi}{\partial III_B} (B^2 - I_B B + II_B I) \right] + \left[ \frac{\partial \phi}{\partial I_B} I_B + \frac{\partial \phi}{\partial II_B} (I_B I - B) + \frac{\partial \phi}{\partial III_B} (B^2 - I_B B + II_B I) \right] \right)
\]
\[
= \frac{1}{\det B^{1/2}} \left[ \frac{\partial \phi}{\partial I_B} B - \frac{\partial \phi}{\partial II_B} B^2 + \frac{\partial \phi}{\partial III_B} I_B B + \frac{\partial \phi}{\partial II_B} B^3 + \frac{\partial \phi}{\partial III_B} II_B B - \frac{\partial \phi}{\partial III_B} I_B B^2 \right]
\]
\[
= \frac{2}{\det B^{1/2}} \left[ \frac{\partial \phi}{\partial III_B} (B^3 + I_B I - B^2) + \left( \frac{\partial \phi}{\partial I_B} + I_B \frac{\partial \phi}{\partial II_B} \right) B - \frac{\partial \phi}{\partial III_B} B^2 \right]
\]

We know from eq. (1.15): \( III_B I = B^3 - I_B B^2 + II_B B \)

So, \( B^2 = I_B B - II_B I + III_B B^{-1} \)

Substituting:
\[
\rightarrow \sigma = \frac{2}{\det B^{1/2}} \left[ III_B \frac{\partial \phi}{\partial I_B} I + \frac{\partial \phi}{\partial II_B} + I_B \frac{\partial \phi}{\partial II_B} B - \frac{\partial \phi}{\partial III_B} I_B B \right]
\]
\[
+ \frac{\partial \phi}{\partial III_B} II_B I - \frac{\partial \phi}{\partial III_B} III_B B^{-1} \right]
\]
\[
= \frac{2}{\det B^{1/2}} \left[ (III_B \frac{\partial \phi}{\partial I_B} I + \frac{\partial \phi}{\partial II_B} I_B I + (\frac{\partial \phi}{\partial I_B} B - (III_B \frac{\partial \phi}{\partial II_B} B^{-1}) B^{-1} \right] \quad (6.9)
\]

Sometimes in other literature (see [2] [23] [20]) \( II_B \) is taken as \( \frac{1}{2}[tr(B^2) - \frac{1}{2}[tr(B)]^2 \) instead of \( \frac{1}{2}[tr(B^2) - tr(B^2)] \). If this is the case, then the last term in eq. (6.9) would be added instead of subtracted, the Cayley-Hamilton Theorem would become \( B^3 - I_B B^2 - II_B B - III_B I = 0 \) instead of \( B^3 - I_B B^2 + II_B B - III_B I = 0 \), and any strain energy functions that are a function of \( II_B \) (see examples to follow) would also need to be modified accordingly.

Incompressibility:
\[ detF = 1(III_B = 1) \quad ; \quad \phi = \phi(I_B, II_B) \]
Modifying eq. (6.8), we get:
\[ \frac{d\phi}{dB} = \left( \frac{\partial \phi}{\partial I_B} + I_B \frac{\partial \phi}{\partial II_B} \right) - \frac{\partial \phi}{\partial II_B} B \]

Substituting this simplified expression into eq. (6.3), we get
\[ \sigma = -\rho_0 I + 2 \left[ \left( \frac{\partial \phi}{\partial I_B} + I_B \frac{\partial \phi}{\partial II_B} \right) B - \frac{\partial \phi}{\partial II_B} B^2 \right] \]  

(6.10)

The underbraced expression is what you would set as the stress for a given deformation. But we can always superimpose an arbitrary pressure (e.g. hydrostatic) to the body without causing deformation. Thus, the basic constitutive expression doesn’t uniquely specify stress \( \sigma \); we introduce \( \rho_0 \) - TBD by boundary conditions.

We could’ve also modified eq. (6.9) with \( II_B = 1 \) and \( \frac{\partial \phi}{\partial III_B} = 0 \) and we would get
\[ \sigma = -\rho_0 I + 2 \left[ \frac{\partial \phi}{\partial II_B} II_B I + \frac{\partial \phi}{\partial I_B} B - \frac{\partial \phi}{\partial II_B} B^{-1} \right] \]  

(6.11)

Then, we can use the Cayley-Hamilton Theorem to substitute for \( B^{-1} \) and arrive at the desired result (eq. (6.10)).

Note, however, that in eq. (6.11), the first term in the square brackets influences all \( \sigma_{ii} \) (diagonal) terms equally just as \( \rho_0 \) does. Thus, this first term can be thought of as another pressure, and can, accordingly, be lumped together with \( \rho_0 \).

So,
\[ \sigma = -\rho_0 I + 2 \left[ \left( \frac{\partial \phi}{\partial I_B} \right) B - \left( \frac{\partial \phi}{\partial II_B} \right) B^{-1} \right] \]  

(6.12)

To convince yourself that eq. (6.12) is correct, go ahead and use either eq. (6.11) or eq. (6.12) in the following Mooney-Rivlin examples. You will see that the results do not change.
6.1 Phenomenological and Micromechanical Models

Hyperelastic constitutive models can be divided into two categories: phenomenological models and micro-mechanical models. Both phenomenological and micro-mechanical models can capture the behavior of a wide range of polymers. Whereas some polymers are compressible, and some polymers can strain elastically an order of magnitude more than others, polymers, generally-speaking, exhibit the same main characteristics, namely, highly nonlinear material behavior. Traditional phenomenological models are not derived from any underlying micro-mechanical physics, but are formed in a manner that seeks to minimize computational and experimental effort while capturing the overall behavior for a particular subset of polymers - e.g. elastomers that stiffen in compression and soften in tension. Three phenomenological models will be presented here: Mooney-Rivlin rubber [24] [28], Blatz-Ko foam [9], and Ogden rubber [26]. Micro-mechanical models, on the other hand, capture some of the underlying physical mechanisms of the materials. Micro-mechanical models have the most potential for improvement, and there are already some micro-mechanical models that can extend to unusual kinds of polymers, where traditional models would diverge from real behavior [8]. Here, only one micro-mechanical model will be presented: Arruda-Boyce rubber [1]. The Arruda-Boyce model, in its original form [1], can handle typical elastomers, but, like any hyperelastic model, will be unable to capture the hyperelastic behavior of all polymers.

Micro-mechanical models may be particularly well suited for “design.” An engineer can easily “create” a micro-mechanical (or phenomenological) polymer model based on, for example, the Arruda-Boyce model, which can be characterized from certain basic material properties. These material properties can be decided by the engineer, and the resulting performance for a particular application can be observed in computer simulations, for example. Once the behavior is satisfactory, according to the simulation, the material can then be manufactured. Micro-mechanical models and some phenomenological models are well suited for such a design process, since they generally depend on basic properties such as initial shear modulus and bulk modulus, and since these are precisely the properties that polymer manufacturers are familiar with.
Mooney-Rivlin e.x. 1

Consider a rectangular block under tensile stress in \( X_1 \) direction (simple extension test) that causes the stretch in that direction of amount \( \lambda_1 \). If the material of the block is an incompressible “Mooney-Rivlin” rubber [24] [28] with:

\[
\phi = \frac{1}{2} \mu \left[ \left( \frac{1}{2} + \beta \right) (I_B - 3) + \left( \frac{1}{2} - \beta \right) (II_B - 3) \right] \tag{6.13}
\]

where \( \mu \) and \( \beta \) are material constants, find the stress required to produce this deformation:

Incompressible \( \rightarrow \sigma = -\rho_0 I + 2 \left[ \left( \frac{\partial \phi}{\partial I_B} \right) B - \left( \frac{\partial \phi}{\partial II_B} \right) B^{-1} \right] \)

\[
\frac{\partial \phi}{\partial I_B} = \frac{1}{4} \mu + \frac{1}{2} \mu \beta ; \quad \frac{\partial \phi}{\partial II_B} = 0 ; \quad \frac{\partial \phi}{\partial III_B} = \frac{1}{4} \mu - \frac{1}{2} \mu \beta
\]

\[
\sigma = -\rho_0 I + 2 \left[ \left( \frac{1}{4} \mu + \frac{1}{2} \mu \beta \right) B - \left( \frac{1}{4} \mu - \frac{1}{2} \mu \beta \right) B^{-1} \right]
\]

We need to figure out a \( x \leftarrow X \) mapping. We could do this by noting that “Poisson Ratio” \( \nu = .5 \) for incompressible materials, but since we have to consider deformation in two transverse directions, it’s better to start with general stretches \( \lambda_1 \) and \( \lambda_2 \), and later use \( \det F = 1 \), as follows:

\[
x_1 = \lambda_1 X_1 \\
x_2 = \lambda_2 X_2 \\
x_3 = \lambda_2 X_3 \quad (\lambda_3 = \lambda_2 \text{ due to isotropy})
\]

\[
F = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_2 \end{bmatrix}
\]

\[
det F = \lambda_1 \lambda_2^2 = 1 \rightarrow \lambda_2 = \frac{1}{\sqrt{\lambda_1}}
\]

\[
\rightarrow B = F \cdot F^T = \begin{bmatrix} \lambda_1^2 & 0 & 0 \\ 0 & \frac{1}{\lambda_1} & 0 \\ 0 & 0 & \frac{1}{\lambda_1} \end{bmatrix}
\]

\[
\sigma_{11} = -\rho_0 + \mu \left[ \left( \frac{1}{2} + \beta \right) \lambda_2^2 - \left( \frac{1}{2} - \beta \right) \frac{1}{\lambda_1} \right]
\]

\[
\sigma_{22} = \sigma_{33} = -\rho_0 + \mu \left[ \left( \frac{1}{2} + \beta \right) \frac{1}{\lambda_1} - \left( \frac{1}{2} - \beta \right) \lambda_1 \right] = 0 \text{ (we know this)}
\]
6.1. PHENOMENOLOGICAL AND MICROMECHANICAL MODELS

\[ \sigma_{11} = \mu \frac{\lambda_1 - 1}{\lambda_1} \left[ \frac{1}{2} + \beta + \left( \frac{1}{2} - \beta \right) \frac{1}{\lambda_1} \right] \]

\(\lambda_1\) is analogous to \(\epsilon_{11}\) (which we’ll see later)

\(\mu, \beta\) are just constants, analogous to \(\mu\), \(\lambda\) (which we’ll see later)

Thus, for this particular Mooney-Rivlin, hyperelastic, incompressible, material, we have a relationship between longitudinal stress and longitudinal strain. This is a method we’ll use later in linear infinitesimal isotropic theory to find the definition of “\(E\)” (Young’s Modulus), for example.

Mooney-Rivlin e.x. 2

Determine the stress and strain state in a rectangular block, made from the Mooney-Rivlin material of e.x. 1, under simple shear of amount \(\varphi\), in the direction \(X_1\):

\[ \sigma = -\rho_0 I + 2\left[ \left( \frac{1}{2} + \beta \right) \frac{\mu}{2} B - \left( \frac{1}{2} - \beta \right) \frac{\mu}{2} B^{-1} \right] \quad \text{(from the previous e.x)} \]

If \(k = \tan \varphi\), \(x_1 = X_1 + kX_2 \); \(x_2 = X_2 \); \(x_3 = X_3 \); \(\sigma_{33} = 0\)

We can easily calculate \(F\) and \(B\):
\[
\mathbf{F} = \begin{bmatrix}
1 & k & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

\[
\mathbf{B} = \mathbf{F} \cdot \mathbf{F}^T = \begin{bmatrix}
k^2 + 1 & k & 0 \\
k & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

\[
\mathbf{B}^{-1} = \begin{bmatrix}
1 & -k & 0 \\
-k & 1 + k^2 & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

\[
\text{note: } \mathbf{E} = \begin{bmatrix}
0 & \frac{1}{2}k & 0 \\
\frac{1}{2}k & \frac{1}{2}k^2 & 0 \\
0 & 0 & 0
\end{bmatrix}
\]

Let’s take a moment to look at Fig. 6.2. Aside from \( \mathbf{B} \) being defined in spatial coordinates, it’s also “Eulerian” and thus is more difficult to interpret, physically, for most engineers. From inspection of \( \mathbf{E} \), however, we can see that the coordinate system must be deformed, as shown in Fig. 6.2, otherwise \( E_{22} \) would be zero.

![Figure 6.2: Deformed axes for Right C-G Tensor](image)

This is mentioned now because the next topic is rate forms, where it is commonly assumed (simplified) that the bases are orthogonal, which can lead to dubious results where large shear deformation is present.

\[
\text{note: both } \sigma_{11} \text{ and } \sigma_{22} \text{ are nonzero but it is common to have stress without strain for an incompressible material}
\]
6.1. PHENOMENOLOGICAL AND MICROMECHANICAL MODELS

\[ \sigma_{11} = -\rho_0 + 2a * \left( \frac{1 + k^2}{B_{11}} \right) - 2b * \left( \frac{1}{[B^{-1}]_{11}} \right) \]

\[ \sigma_{22} = -\rho_0 + 2a * \left( \frac{1}{B_{22}} \right) - 2b * \left( \frac{1 + k^2}{[B^{-1}]_{22}} \right) \]

\[ \sigma_{12} = 2(a + b)k \quad \text{no } \rho_0 \text{ because } \rho I_{12} = 0 \text{ due to I being a diagonal matrix} \]

\[ \sigma_{33} = -\rho_0 + 2a - 2b \]

\[ \sigma_{23} = 0 \]

\[ \sigma_{31} = 0 \]

\[ \sigma_{33} = 0 \rightarrow -\rho_0 + 2a - 2b = 0 \rightarrow \rho_0 = 2(a - b) \]

( substitute into the above expressions)

\[ \sigma_{11} = 2ak^2 \quad \sigma_{22} = -2bk^2 \quad \sigma_{12} = 2(a + b)k \]

\[ \text{note: } k = \tan \phi = \varphi \text{ for infinitesimal strains } \rightarrow \varphi^2 << \varphi \rightarrow \sigma_{11} \text{ and } \sigma_{22} \text{ may become negligible} \]

\[ \text{note: } \mathbf{E} \text{ and } \mathbf{e} \text{ both approach } \begin{bmatrix} 0 & k & 0 \\ k & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ for infinitesimal strain, but they would not be equal if large rigid body rotations were present} \]

Examples 1 and 2 derive the necessary formulas to determine the basic material properties that fully describe a Mooney-Rivlin rubber. For example, an axial test would give you \( \sigma_{11} \) and \( \lambda_1 \) while a shear test would give you \( \sigma_{12} \) and \( "k." \) These would be unique for a particular Mooney-Rivlin rubber. Once these data are known, then there are two equations in \( \sigma_{11}, \sigma_{12} \) and one can solve for the two unknowns: \( \mu \) and \( \beta \).

Blatz-Ko Foam

Consider the following strain energy function, which is a simplified version of the Blatz-Ko function \[9\]:

\[ \phi = \frac{1}{2} \mu \left( 2III^{1/2}_B + II_BIII^{-1}_B - 5 \right) \]

We can derive the constitutive equation for \( \sigma \) in terms of \( \mu \) and \( \mathbf{B} \) as follows:
\[ \frac{\partial \phi}{\partial I_B} = 0 \quad \frac{\partial \phi}{\partial II_B} = \frac{1}{2} \mu III_B^{-1} \quad \frac{\partial \phi}{\partial III_B} = \frac{1}{2} \mu (III_B^{-1/2} - II_B III_B^{-2}) \]

Substituting into eq. (6.9), re-written below:

\[ \sigma = \frac{2}{\det B^{1/2}} \left[ (III_B \frac{\partial \phi}{\partial III_B} + \frac{\partial \phi}{\partial II_B} II_B) I + \frac{\partial \phi}{\partial I_B} B - III_B \frac{\partial \phi}{\partial II_B} B^{-1} \right] \]

We then get:

\[ \sigma = \frac{2}{\det B^{1/2}} \left[ (III_B * \frac{1}{2} \mu (III_B^{-1/2} - II_B III_B^{-2}) + \frac{1}{2} \mu III_B^{-1} I) I \\
-III_B * \frac{1}{2} \mu III_B^{-1} B^{-1} \right] \]

This reduces to:

\[ \sigma = \frac{1}{\det B^{1/2}} * \mu (III_B^{1/2} I - B^{-1}) \]

or

\[ \sigma = \mu \left( I - III_B^{-1/2} B^{-1} \right) \]

\( \mu \) is analogous to the “shear modulus” as we’ll see in linear infinitesimal elasticity in a later chapter. However, in order to simplify experimentation, a simple uniaxial test is often performed in order to determine \( \mu \) for materials that follow the above Blatz-Ko function.

Arruda-Boyce Rubber

Before we get to the Ogden rubber model, which will be discussed at length in this section and the next one, let’s briefly describe the Arruda-Boyce model [1]. The Arruda-Boyce model, sometimes called the “Eight-Chain model” is a very popular model and, along with the Ogden model, is a function of the so-called “principal stretches.” It is a statistically derived model based on the macromolecular physical mechanics of “long-chain” materials.
6.1. **PHENOMENOLOGICAL AND MICROMECHANICAL MODELS**

The strain energy density function is given as:

\[
\phi = nk\Theta \left[ \frac{1}{2} (I_B - 3) + \frac{1}{20N} (I_B^2 - 9) + \frac{11}{1050N^2} (I_B^3 - 27) \right] + nk\Theta \left[ \frac{19}{7000N^3} (I_B^4 - 81) + \frac{519}{673750N^4} (I_B^5 - 243) \right] + \frac{K}{2} (III_B - 1)^2
\]

\(nk\Theta\) are, respectively, the “chain density,” Boltzmann’s constant, and temperature, and as a product represent the shear modulus, \(G\).

Note that the last term in the Arruda-Boyce strain energy density function goes to zero for incompressible (“rubbery”) material. We can also see that the Arruda-Boyce rubber model depends only on the principal stretches - i.e. we can substitute \(I_B = \lambda_1^2 + \lambda_2^2 + \lambda_3^2\) into the above equation, where \(\lambda_1\), \(\lambda_2\), and \(\lambda_3\) are the eigenvalues of the Left Stretch Tensor, \(V\), or the Right Stretch Tensor, \(U\). Recall that \(V\) and \(U\) have the same eigenvalues and the same invariants, but different eigenvectors. The \(\lambda\) are commonly called the principal stretches, and usually refer to the eigenvalues along the principal axes of \(V\). To find a constitutive relationship for the Cauchy Stress, \(\sigma\), we could use the following relationship (derived in Appendix C.3):

\[
\sigma_i = \frac{1}{\lambda_j \lambda_k} \frac{\partial \phi}{\partial \lambda_i} \text{ no sum}
\]

\(\sigma_i\) are the principal true stresses (the eigenvalues of the Cauchy Stress, \(\sigma\)).

Alternatively, for incompressibility, and where \(\phi\) is only a function of \(I_B\), we have:

\[
\sigma_i = -p + \frac{1}{IIH_N} \lambda_i \frac{\partial \phi}{\partial I_B} \frac{\partial I_B}{\partial \lambda_i}
\]

where “\(p\)” is our “hydrostatic” term that would need to be found from boundary conditions.

Since \(\frac{dI_B}{d\lambda_i}\) is \(2\lambda_i\), we get:

\[
\sigma_i = -p + 2\lambda_i^2 \frac{\partial \phi}{\partial I_B} \quad (6.14)
\]

Keep in mind that eq. (6.14) is only valid if \(\phi = \phi(I_B)\).
The Arruda-Boyce model presented here is the original version [1], and is typically used to describe soft elastomers at large strains. Other versions of this model exist [7] that are more suitable for describing the behavior of elastomers in the smaller strain range (less than 30%). This kind of behavior is most often of interest for relatively stiff polymers.

That’s all that will be said about the Arruda-Boyce model. What we will do in the remainder of this section is derive the constitutive expression for the Ogden model, which we will need in order to form our “tabulated” constitutive relationship in the next section, which will be based on the Ogden model.

Ogden Rubber

The strain energy density function for an Ogden material [26], is given by

\[ \phi = \sum_{n=1}^{3} \sum_{s=1}^{m} \frac{\mu_s}{\alpha_s} (\lambda_n^{*\alpha_s} - 1) + K(\text{III}_V - 1 - \ln \text{III}_V) \]  

(6.15)

In eq. (6.15), \( \mu_s \) and \( \alpha_s \) are material constants.

\( \lambda_n^{*} = \lambda_n * \text{III}_V^{-1/3} \) and \( \text{III}_V = \lambda_1 \lambda_2 \lambda_3 \)

Eq. (6.15) is valid for nearly incompressible materials.

The strain energy density function, \( \phi \), for the Ogden model will be a function of the principal stretches, just as in the Arruda-Boyce model. We will make use of the constitutive relationship:

\[ \sigma_i = \frac{1}{\lambda_j \lambda_k} \frac{\partial \phi}{\partial \lambda_i} \]  

(6.16)

Eq. (6.16) is derived in Appendix C.3.

To derive the Ogden constitutive relationship, we need to find \( \frac{\partial \phi}{\partial \lambda_i} \) for the Ogden strain energy density function.

In the following derivation, \( i, j, k \), will be used as subscripts to indicate eigenvalues 1, 2, 3. Unless indices are repeated, they should not be summed.
6.2 Tabulated and Calibrated Models

In the past, experimental constants would typically be found by hand. Given a known material, it can be time-consuming to identify their parameters, even after the necessary experimental tests have been performed - e.g. since hyperelastic materials are nonlinear by definition, matching the experimental data is not so trivial as finding the slope of a uniaxial stress-strain curve, as would be the case for a linear material. The Mooney-Rivlin example in the last section showed that the procedure for obtaining experimental constants can be cumbersome. For more advanced hyperelastic models, finding the experimental constants would be even more difficult. More recently, two methods have been developed to “fit” a particular material to a particular hyperelastic model - the “tabulated” method and the “calibration” method.
Before we look at a particular tabulated model as well as a particular method for calibrating a model, consider that for many hyperelastic models, uniaxial data is sufficient to fully characterize their behavior. Exceptions would include models that are heavily influenced by higher order strain invariants. Regardless, uniaxial data is the most common kind of experimental data that is used for characterizing materials - hyperelastic materials included. In fact, both of the following examples will fit a hyperelastic model to uniaxial data, at the exclusion of any other kind of experiment data.

These characterization methods may be considered “curve-fitting” techniques, but keep in mind that the hyperelastic models that we are attempting to match to a uniaxial curve are tensorial, not scalar-valued. Using a tabulated approach, the challenge is to take a hyperelastic function, which forms a work-conjugate constitutive relationship that naturally extends to a 3D environment, and modify it to “grab” data from tabulated uniaxial stress-strain data. A particular tabulated approach developed by Stefan Kolling of Daimler AG, in collaboration with P.A. Du Bois of LSTC and David Benson of UCSD [17], will be presented, which is based on the Ogden model and generates a “perfect fit.”

\[
f_0(\lambda_i) = \sum_{x=0}^{\infty} \lambda_i^{(-1/2)x} \sigma_0 \left( \lambda_i^{(-1/2)x} - 1 \right) \tag{6.20}
\]

\[
\sigma_i = \frac{1}{III} \left( f_0(\lambda_i) - \frac{1}{3} \sum_{n=1}^{3} f_0(\lambda_n) \right) + K \frac{1}{III} - 1 \tag{6.21}
\]

The complete derivation of eq. (6.20) and eq. (6.21) can be found in Appendix C.4, based on the descriptions provided in [17].

In a simplified FEM algorithm (no rates/hypoelasticity), the series expressed above (eq. (6.20)), which calculates a single value of \( f_0(\lambda_i) \) from the uniaxial data by summing (over “x”) to \( \infty \), would be terminated once a reasonable tolerance is met. In addition, a sufficient number of \( f_0(\lambda_i) \) values could be calculated, where each value is calculated from eq. (6.20) and corresponds to a particular value of \( \lambda_i \). The number of \( f_0(\lambda_i) \) values could perhaps be equal to the number of \( \sigma_0(\epsilon_{0i}) \) values that have been provided by the user, presumably from a uniaxial engineering stress vs. engineering strain plot. Note that all such \( f_0(\lambda_i) \) values can be found during the initialization of the
6.2. **TABULATED AND CALIBRATED MODELS**

FEM problem, prior to the start of the simulation.

In solving actual problems using this material model (though this is still only for illustrative purposes, since real problems involve time and necessitate a hypoelastic constitutive formulation), the actual stretches are the eigenvalues of $V$ and would be determined from the actual loading. $\sigma_1$, $\sigma_2$, $\sigma_3$, which are the eigenvalues of the Cauchy stress, $\sigma$, can be calculated from eq. (6.21), where “$i$” in eq. (6.21) refers to the eigenvalue 1, 2, or 3. Obtaining the Cauchy stress, $\sigma$, from its eigenvalues and eigenvectors involves just a single transformation.

While the original form of the Ogden model (eq. (6.19)) contains an arbitrarily large number of constants, it is not necessarily true that the Ogden model, in its original form, can fit any uniaxial stress vs. strain curve. In its original form, the Ogden model contains the constants $\mu_s$ and $\alpha_s$, and these constants (of which there exist $2m$ of them) would be solved simultaneously. One possible way to attempt this would be to start with eq. (6.20), where we recall that the left-hand-side of eq. (6.20) is $\sum_{s=1}^{m} \mu_s \lambda_i^{\mu_s} = f_0(\lambda_i)$. It is not clear that creating multiple equations from eq. (6.20) and solving simultaneously for these constants is feasible.

The method presented, on the other hand, which is referred to as a “tabulated” method, can fit any uniaxial curve, perfectly. One can consider that this tabulated method is possible because of a number of particular characteristics in the original Ogden expression (eq. (6.19)), and clever “tricks” employed by the authors of the tabulated model [17], which are shown in Appendix C.4.

The value of this “tabulated” method cannot be understated: it makes possible the exact fit to any uniaxial stress vs. strain experimental data. The material would then be characterized by Ogden hyperelasticity, which, in theory should work quite well for any type of loading condition that the material may see (torsion, bending, etc.). Viscoelasticity would be the primary concern, and this will be briefly discussed at the end of this chapter.

A “calibration” method will be shown next, and then the two methods will be briefly compared. Fig. 6.3 is a screen-shot from a software called MCalibration, developed by Jorgen Bergstrom of Veryst Engineering®. MCalibration contains a library of materials called PolyUMod [7]. Contained within this library are many hyperelastic materials, including the Arruda-Boyce,
Mooney-Rivlin, and Ogden.

Data from a uniaxial compression experiment was loaded into MCalibration. We can assume, here, that the compression data represents the relaxed equilibrium response of the material - in other words, it is hyperelastic. The Yeoh Model [35] was then calibrated, since this model provided a better fit, for this particular data set, compared to the Arruda-Boyce or the Ogden model. We can see that the fit is still not perfect.

The advantage of the tabulated method, as stated previously, is that it will match any uniaxial curve, exactly, and take almost no computational effort to do so. A disadvantage of the tabulated method is that physical material properties never enter the conversation, so error in the uniaxial data can go unrecognized. Still, there are surprisingly very few tabulated models, like the one that has been presented, which are built from hyperelastic functions. This may be due to concerns related to the treatment of viscoelasticity, as will be briefly discussed.
Viscoelasticity

Perhaps the real reason that we don’t see more tabulated models is because for many kinds of polymers and for many applications that involve polymers, viscoelastic rate effects on loading and unloading behavior are as important as the underlying hyperelastic component of behavior. In reality, it is extremely difficult to obtain the relaxed equilibrium response of an elastomer, and so even a “quasi-static” uniaxial test will have different loading and unloading response. In other words, hyperelasticity is only one aspect of the response of an elastomer under realistic loading rates.

The specific implementation of the tabulated model previously described [17] is the LS-DYNA material model called:
*MAT_SIMPLIFIED_RUBBER_WITH_DAMAGE [13]

While this model can handle a user-defined uniaxial loading curve as well as an unloading curve, it does not handle hyper-viscoelasticity in a physical sense. This is why the developers of the model [17] use the word “simplified” and “damage” to describe the hysteretic behavior capability. No tabulated method, to the author’s knowledge, exists that can handle both hyperelasticity and viscoelasticity in a physical sense. The PolyUMod library, however, consists of many viscoelastic options, and MCalibration is capable of “fitting” uniaxial hysteretic data using models that contain components of both hyperelasticity and viscoelasticity. This topic, however, is beyond the scope of this text.
Chapter 7

Rate-Form Constitutive Expressions

We are going to begin discussion on “hypoelasticity”, which deals with rates of stress and strain. In FEM, this permits the treatment of dynamic problems and history-dependent problems, and thus is the basic way that advanced computer software develop constitutive relationships. As always, it is important to recognize which measures of stress and strain are invariant to rigid body rotation and which are not. Whether we are talking about hyperelasticity or linear infinitesimal elasticity, it is always easy to make up a problem in which a body deforms in some prescribed fashion, with large rigid body rotations, and then determine how measures of strain behave under such deformations and rigid body rotations. This is relatively simple to do since strain is a direct function of deformation (e.g. $\mathbf{E}$ is a direct function of $\mathbf{F}$), and so this strain behavior can essentially be seen. For example, the Lagrangian strain tensor, $\mathbf{E}$ (as well as the linear infinitesimal strain tensor, $\mathbf{\epsilon}$), does not change with rigid body rotation. This was shown via example in previous chapters.

Stress is not so easy to “see.” The Cauchy stress tensor, $\mathbf{\sigma}$, which is used in hyperelasticity, does change with rigid body rotation (it is measured in “spatial” coordinates rather than “material” coordinates). The linear infinitesimal stress tensor, as it turns out (next chapter), can be thought of as $\mathbf{\hat{\sigma}}$ (recall eq. (4.6)) in the sense that the linear infinitesimal stress tensor is defined in the same coordinate system as the strain tensor - it is invariant to rigid body rotation. Tensors that behave similarly under rigid body rotation (e.g. $\mathbf{\hat{\sigma}} \leftrightarrow \mathbf{E}, \mathbf{\sigma} \leftrightarrow \mathbf{D, B}$) are said to be “work-conjugate.” In order to form a
Chapter 7. Rate-Form Constitutive Expressions

Proper constitutive relationship, the stress and strain tensors involved must be work-conjugate.

Recall the e.x. starting on pg. 38.

Fig. 7.1 shows a simpler version of this example:

Prior to the rigid body rotation (i.e. at time \( t = t_0 \)):

\[
\sigma = \begin{bmatrix} \sigma_{11} & 0 \\ 0 & 0 \end{bmatrix}
\]

After this 90° rigid body rotation (at time \( t = t_f \)), there are two possibilities:

\[
\sigma = \begin{bmatrix} \sigma_{11} & 0 \\ 0 & 0 \end{bmatrix}
\] (7.1)

or

\[
\sigma = \begin{bmatrix} 0 & 0 \\ 0 & \sigma_{11} \end{bmatrix}
\] (7.2)

Unless we perform some mathematical “tricks” between \( t_0 \) and \( t_f \), tensor (7.1) would be the tensor for linear infinitesimal elasticity, \( \dot{\sigma} \), at time \( t_f \),
and tensor (7.2) would be the Cauchy stress tensor, $\sigma$, at time $t_f$.

In some literature it is stated that the linear infinitesimal stress tensor is the Cauchy stress and the linear infinitesimal strain tensor corresponds to the Eulerian strain, $e$. We have seen (example that started on page 38), the physical interpretation of the Eulerian strain, and the choice of strain measure for the case of linear infinitesimal elasticity is essentially up to the individual. This can be a source of confusion since $E$ is not equal to $e$ when rigid body rotations are present, but this will be addressed at the end of this chapter when we get to our “time-stepping algorithm.”

In real-life structural analysis simulations, adjacent elements need to have a consistent frame of reference. As described in [30], under nonhomogeneous deformations, adjacent elements will not “fit together” if we use the material or “local” configuration (eq. (7.1)). With this in mind, eq. (7.2) (i.e. a “spatial” representation), is what we need.

We’ll first consider the rate form of Cauchy stress, $\sigma$, before taking a closer look at the infinitesimal stress, $\hat{\sigma}$. One reason that finite hyperelasticity (nonlinear) theory and linear infinitesimal theory typically use different work-conjugate stress/strain pairs is because they were developed independently [23]. We will thus be required to keep track of numerous different measures of stress and strain, which may benefit the reader or may be viewed as a nuisance, depending on perspective.

note: In this text, the word “spatial” refers to the coordinate system at time zero. This chapter will conclude with the presentation of one possible (and very simple) “time-stepping algorithm.”

### 7.1 Hypoelasticity (Jaumann)

Recall that $\sigma$ and $B$ were shown to be work-conjugate, and this permitted us to develop a constitutive relationship between the two tensors. Now we would like to develop a constitutive relationship in hypoelastic form. Thus, we need a stress rate and a strain rate that are work-conjugate. Unfortunately, the time derivative of the Cauchy stress tensor, $\dot{\sigma}$, is not work-conjugate with $D$, where we recall that $D$ is the “rate of deformation tensor.” Consider the following:
Here, $\bar{\nabla}$ is the “Truesdell rate.” Essentially, we wrote $\sigma$ in terms of the Second Piola-Kirchhoff Stress, and then, within this expression, we took the time derivative of only the Second Piola-Kirchhoff Stress (recall from eq. (4.6), that the Second-Piola Kirchhoff Stress, $\dot{\sigma}$, is $(det F) F^{-1} \cdot \sigma \cdot F^{-T}$).

The time derivative expressed in eq. (7.3) can be taken, which requires the use of the product rule for the four terms within the square brackets. An alternative derivation is given in Appendix D.2. In either case, eq. (7.3) becomes:

$$\bar{\nabla} = \dot{\sigma} - L \cdot \sigma - \sigma \cdot L^T + tr(L)\sigma$$  \hspace{1cm} (7.4)

Recall the definition of the velocity gradient, $L$, from eq. (3.1). Additionally, note that $trL = trD$.

Eq. (7.4) is the most commonly used form of the Truesdell rate. In eq. (7.4), if we eliminate all of the $D$ terms except for those that appear in the constitutive relationship (we can also think of this simplification as taking $L \approx \dot{R} \cdot R^T \approx W$), then we arrive at the following result:

$$\dot{\sigma} = \dot{\sigma} - W \cdot \sigma - \sigma \cdot W^T$$  \hspace{1cm} (7.5)

or

$$\dot{\sigma} = \dot{\sigma} - W \cdot \sigma + \sigma \cdot W$$  \hspace{1cm} (7.6)

\textbf{note: }$W^T = -W$

In hypoelasticity, eq. (7.5) is called the Jaumann rate of Cauchy Stress. We will form our hypoelastic constitutive relationship in terms of this rate. Then, we’ll see how this constitutive relationship would be used to solve real FEM problems, through a simple three step time-stepping algorithm. But first, we need to prove that the Jaumann rate of Cauchy Stress is indeed
work conjugate with \( D \) - i.e. does \( \dot{\sigma}^* = Q \cdot \dot{\sigma} \cdot Q^T \)?

Note that \( \dot{\sigma} \) can represent the linear infinitesimal stress. It turns out that most FEM software use the Jaumann rate for linear infinitesimal stress as well. The reason for this will be explained at the end of this chapter, along with a brief discussion of linear infinitesimal elasticity time-stepping.

Recall \( \sigma^* = Q \cdot \sigma \cdot Q^T \)

\[
\dot{\sigma}^* = \dot{Q} \cdot Q^{-1} \cdot \sigma^* + Q \cdot \dot{\sigma} \cdot Q^T + Q \cdot \sigma \cdot \dot{Q}^T \cdot Q \cdot \dot{Q}^T
\]

\[
= \dot{Q} \cdot Q^{-1} \cdot \sigma^* + Q \cdot \dot{\sigma} \cdot Q^T + (\dot{Q} \cdot Q^{-1} \cdot Q \cdot \sigma \cdot Q^T) \cdot Q \cdot \dot{Q}^T
\]

\[
\rightarrow \dot{\sigma}^* = (W^* - Q \cdot W \cdot Q^T) \cdot \sigma^* + Q \cdot \dot{\sigma} \cdot Q^T + \sigma^* \cdot (\dot{Q} \cdot Q^{-1} \cdot Q \cdot W \cdot Q^T) \cdot Q \cdot \dot{Q}^T
\]

Since \( W^* = \dot{Q} \cdot Q^{-1} + Q \cdot W \cdot Q^T \rightarrow \dot{Q} \cdot Q^{-1} = W^* - Q \cdot W \cdot Q^T \)

\[
\rightarrow \dot{\sigma}^* - W^* \cdot \sigma^* + \sigma^* \cdot W^* = Q \cdot \dot{\sigma} \cdot Q^T + \sigma^* \cdot Q \cdot W \cdot Q^T - Q \cdot W \cdot Q^T \cdot \sigma^* \cdot (Q \cdot Q^{-1} \cdot Q \cdot W \cdot Q^T) \cdot Q \cdot \dot{Q}^T
\]

Moving some of the terms to the other side:

\[
\rightarrow \dot{\sigma}^* - W^* \cdot \sigma^* + \sigma^* \cdot W^* = Q \cdot (\dot{\sigma} - W \cdot \sigma + \sigma \cdot W) \cdot Q^T
\]

Since we know that \( \dot{\sigma} = \dot{\sigma} - W \cdot \sigma + \sigma \cdot W \)

\[
\rightarrow \dot{\sigma}^* = Q \cdot \dot{\sigma} \cdot Q^T \quad (7.7)
\]

Note the similarity to \( \sigma^* = Q \cdot \sigma \cdot Q^T \) and \( D^* = Q \cdot D \cdot Q^T \) (this is what we wanted to show)

Now, just as we have derived \( \sigma \leftrightarrow B \) in the past, we can now derive \( \dot{\sigma} \leftrightarrow D \) (\( \dot{\sigma} \) will also depend on \( \sigma, B \), and \( B \)).

Truesdell et al. [32] coined the phrase “hypoelasticity.” All hyperelastic constitutive relationships can be put into rate form (and hence are hypoelastic). The reverse is not always possible [30]. Materials that are viscoelastic, or any material that is expected to experience plasticity, can be conveniently expressed in rate form, where history dependent variables can be tracked,
naturally.

\[
\begin{align*}
\mathbf{B}^* &= \mathbf{Q} \cdot \mathbf{B} \cdot \mathbf{Q}^T \\
\sigma^* &= \mathbf{Q} \cdot \sigma \cdot \mathbf{Q}^T \\
\mathbf{D}^* &= \mathbf{Q} \cdot \mathbf{D} \cdot \mathbf{Q}^T \\
\dot{\sigma}^* &= \mathbf{Q} \cdot \dot{\sigma} \cdot \mathbf{Q}^T
\end{align*}
\]

Since all of the tensors above are work-conjugate, it is convenient to use them to derive our final form of \( \dot{\sigma} \).

Recall \( \sigma = \frac{1}{\det \mathbf{B}} (\mathbf{B} \frac{\partial \phi}{\partial \mathbf{B}} + \frac{\partial \phi}{\partial \mathbf{B}} \mathbf{B}) \):

We will form our hypoelastic constitutive relationship by applying the Jaumann "operator" (\( \cdot \)) to both sides of this expression. But, before we do this, we ought to find \( \dot{\det \mathbf{F}} \) and \( \dot{\mathbf{B}} \).

\( \det \mathbf{F} = \det \mathbf{F} \) (i.e. presumably, the analog to \( -\mathbf{W} \cdot \mathbf{B} + \mathbf{B} \cdot \mathbf{W} \) would simply be zero, since any asymmetries cancel out when taking the determinant)

From eq. (2.9) we know that \( dV = dV_0 \det \mathbf{F} \) and from eq. (3.5) we know that \( \frac{d}{dt} (dV) = D_{kk} dV \).

So, \( \frac{d}{dt} dV = D_{kk} dV = \dot{\det \mathbf{F}} dV_0 \). Thus,

\[
\dot{\det \mathbf{F}} = \frac{dV}{dV_0} D_{kk} = (\det \mathbf{F}) tr \mathbf{D} \quad (7.8)
\]

In terms of \( \mathbf{B} \): \( \det \mathbf{F} = \det \mathbf{B}^{1/2} \rightarrow \frac{d}{dt} \det \mathbf{B}^{1/2} = \det \mathbf{F} tr \mathbf{D} \)

Next, we need to find \( \mathbf{B} \):

\[
\mathbf{B} = \mathbf{F} \cdot \mathbf{F}^T \\
\rightarrow \dot{\mathbf{B}} = \dot{\mathbf{F}} \cdot \mathbf{F}^T + \mathbf{F} \cdot \dot{\mathbf{F}}^T = \left( \mathbf{F} \cdot \mathbf{F}^{-1} \right) \cdot \mathbf{F} \cdot \mathbf{F}^T + \mathbf{F} \cdot \mathbf{F}^T \cdot \mathbf{F}^{-T} \cdot \dot{\mathbf{F}}^T = \mathbf{L} \cdot \mathbf{B} + \mathbf{B} \cdot \mathbf{L}^T
\]

\[
\dot{\mathbf{B}} = \mathbf{L} \cdot \mathbf{B} + \mathbf{B} \cdot \mathbf{L}^T = \mathbf{D} \cdot \mathbf{B} + \mathbf{B} \cdot \mathbf{D} + \mathbf{W} \cdot \mathbf{B} - \mathbf{B} \cdot \mathbf{W}, \quad (\text{since } \mathbf{L} = \mathbf{D} + \mathbf{W} \text{ and } \mathbf{L}^T = \mathbf{D}^T + \mathbf{W}^T = \mathbf{D} - \mathbf{W})
\]
\[ \dot{B} = (D \cdot B + B \cdot D + W \cdot B - B \cdot W) - W \cdot B + B \cdot W = D \cdot B + B \cdot D \]

Reminder: The direct calculation of the time derivative of \( B \) (i.e. \( \dot{B} \)), similar to \( \dot{\sigma} \), is not work-conjugate with \( D \). This is why we need to use \( \dot{B} \), just as we are using \( \dot{\sigma} \).

\[ \det B^{1/2} \sigma = B \frac{d\phi}{dB} + \frac{d\phi}{dB} B \quad \text{apply the Jaumann operator (\( \dot{\cdot} \)) to this entire expression.} \]

Noting that \( B \frac{d\phi}{dB} = \dot{B} \frac{d\phi}{dB} + B \frac{d^2\phi}{dB^2} : \dot{B} \)

\[ \rightarrow \sigma \det B^{1/2} tr D + \det B^{1/2} \sigma = \dot{B} \frac{d\phi}{dB} + B \frac{d^2\phi}{dB^2} : \dot{B} + (\frac{d^2\phi}{dB^2} : \dot{B}) \cdot B + \frac{d\phi}{dB} \dot{B} \]

note: Recall from Section 1.4 that \( A : B = tr(A \cdot B^T) = A_{ij}B_{ij} \)

So, \( \dot{\sigma}_{ij} = -\sigma_{ij}D_{kk} + A_{ijmn} \dot{B}_{mn} \)

where (skipped work):

\[ A_{ijmn} = \frac{1}{\det B^{1/2}} [\delta_{im} \frac{d\phi}{dB_{nj}} + \frac{d\phi}{dB_{nm}} \delta_{jn} + B_{ik} \frac{d^2\phi}{dB_{kj}dB_{mn}} + \frac{d^2\phi}{dB_{ik}dB_{mn}} B_{kj}] \]

Since \( \dot{B} = B \cdot D + D \cdot B \), all terms in the above expression involve \( D \). Thus, we arrive at the following relationship:

\[ \dot{\sigma}_{ij} = \Lambda_{ijkl}D_{kl} \quad (7.9) \]

Eq. (7.9) is the Jaumann rate of Cauchy Stress in terms of deformation, stress, and rate of deformation.

Note that we could’ve formed a slightly different hypoelastic relationship using the Truesdell rate. Either way, the hypoelastic expression is going to be complicated (the Truesdell rate, slightly more so). This is, however, meant to be broad enough to treat materials that are both nonlinear elastic, and subjected to large strain.
7.2 Time Stepping Algorithm

In hypoelasticity, we required the use of the so-called “Jaumann rate of Cauchy Stress”, \( \dot{\sigma} \), because \( \dot{\sigma} \) is not work conjugate with \( \sigma \) or \( D \), while \( \dot{\sigma} \) is. However, we still need \( \dot{\sigma} \), because \( \sigma_n = \sigma_{n+1} + \dot{\sigma} \Delta t \) is the stress tensor that we need. The Cauchy stress (and its rate) have a well-understood physical meaning (whereas \( \dot{\sigma} \) and \( \dot{\sigma} \Delta t \) do not have physical meaning) - namely, \( \sigma \) is defined in the “spatial” coordinate system. Recall once more that this is necessary in order to have a consistent frame of reference for all of the elements in the finite element simulation. Thus, before we can consider the problem “solved” for any particular instance in time, we need to add an additional “step” to our solution procedure that obtains \( \dot{\sigma} \) from \( \dot{\sigma} \). We can do this by simply solving for \( \dot{\sigma} \) from our previous expression for the Jaumann rate of Cauchy Stress (eq. (7.5)). Thus, during each time step, the Jaumann rate is the stress that is used for the constitutive relationship, but the Cauchy stress is what is needed for the equation of motion (e.x. stress equilibrium).

It turns out that this same multi-step time-stepping “algorithm” will work for linear infinitesimal elasticity as well. We already mentioned that \( \dot{\sigma} \) is analogous to the linear infinitesimal stress tensor, and \( E \) is analogous to the linear infinitesimal strain tensor, \( \epsilon \). In other words, the work-conjugate pair - defined in material coordinates - that is used in linear infinitesimal elasticity, can be thought of as a simplified version of \( \dot{\sigma} \leftrightarrow E \). \( \dot{\sigma} \) and \( \dot{E} \) we know to be similarly work-conjugate and invariant to rigid body rotation.

Thus, one possible expression in linear infinitesimal elasticity that is analogous to our hypoelastic expression (eq. (7.9)) could be:

\[
\dot{\sigma}_{ij} = C_{ijkl} \epsilon_{kl}
\]

where \( C_{ijkl} \) is a fourth-order tensor that relates linear infinitesimal stress, \( \dot{\sigma} \), to linear infinitesimal strain, \( \epsilon \), as we will see in the next chapter.

Obtaining the stress in spatial coordinates can be accomplished via eq. (4.6), namely, \( \dot{\sigma} = \frac{d}{dt} \left[ (det F) F^{-1} \cdot \sigma \cdot F^{-T} \right] \). After performing this time derivative, we could then rearrange to find the rate of Cauchy stress, \( \dot{\sigma} \), in terms of \( \dot{\sigma} \). However, it may be desirable to take a different approach in order to avoid having to explicitly calculate the deformation gradient, \( F \), or its time-derivative \( \dot{F} \).
In addition, we might want to avoid material coordinates (e.g., we might want to avoid calculating $\dot{\epsilon} \approx \dot{E}$, even though $E$ is the most intuitive measure of strain). As we’ll see shortly, in our particular “time-stepping algorithm,” at no time do we consider the material reference frame. This is also one reason why we did not form our finite-strain hypoelastic relationship in the form $\dot{\sigma} = f(E, \dot{E})$.

Let’s instead consider our previous definitions of the Truesdell and the Jaumann rate, which, for infinitesimal elasticity, state that:

$$\nabla \sigma \approx \sigma \approx F \cdot \dot{\sigma} \cdot F^T$$

Here, we should note that the term $\frac{1}{\det F}$ was eliminated since $\det F \approx 1$ for infinitesimal deformations.

Since $\dot{\sigma} = C : \dot{e}$ and $\dot{\sigma} \approx F \cdot \dot{\sigma} \cdot F^T$, we can see that:

$$\dot{\sigma} = C : (F \cdot \dot{e} \cdot F^T)$$

From Chapter 3, we know that eq. (7.10) conveniently reduces to:

$$\dot{\sigma} = C : D$$

Or,

$$\dot{\sigma}_{ij} = C_{ijkl} D_{kl}$$

In eq. (7.11), we recall that $D$ is the “Rate of Deformation Tensor.”

Some authors write the constitutive relationship for linear infinitesimal elasticity as $\sigma = C : e$. The physical nature of $E$ and $e$ was illustrated in the example that started on page 38, where we saw that the two strain measures gave different results, even for the infinitesimal deformation case. Ultimately, however, one can argue that we are free to choose either stress-strain pair in linear infinitesimal elasticity. Applying the Jaumann operator to either constitutive expression results in the equation (7.11). Thus, our “time-stepping algorithm,” which we will see shortly, is independent of our “interpretation” of linear infinitesimal stress and strain.
The Kirchhoff stress, $\tau$, is defined as $\tau = (\det F)\sigma$. Sometimes $\tau$ is used in place of $\sigma$, in eq. (7.11) for example, so that the small deformation approximation of $\det F \approx 1$ does not need to be used.

Obtaining the stress rate in spatial coordinates simply requires that we solve for $\dot{\sigma}$ from eq. (7.5), namely,

$$\dot{\sigma} = \dot{\sigma} + W \cdot \sigma + \sigma \cdot W^T$$

(7.13)

An alternative proof of eq. (7.13), for infinitesimal elasticity, is given in Appendix D.1

Thus, our constitutive expressions, in rate-form, for both linear infinitesimal elasticity and hyperelasticity (hypoelasticity), involve the Jaumann rate. For hypoelasticity, we had eq. (7.9), while for linear infinitesimal elasticity, we have eq. (7.12). The following time-stepping algorithm, which allows us to obtain Cauchy (“spatial”) stress is identical for both hyperelasticity and linear infinitesimal elasticity.

note that the bases do not always remain orthogonal when there is shear (recall Fig. 6.2, for example). The Jaumann rate in eq. (7.5) assumes orthogonal bases, which is a simplification.

General Framework For Problem Solving

1. Use an appropriate work-conjugate constitutive relationship:
   $$\dot{\sigma}_{ij} = \Lambda_{ijkl}D_{kl} \text{ or } \dot{\sigma}_{ij} = C_{ijkl}D_{kl}$$

2. Obtain a spatial representation of stress by first finding $\dot{\sigma}$ from eq. (7.13), and then finding $\sigma$ from $\sigma_{n+1} = \sigma_n + \dot{\sigma} \Delta t$

3. Solve the equation of motion for the next increment in time:
   $$\sigma_{ij,j} + \rho b_i = \rho \frac{dv_i}{dt}$$

The majority of this text is devoted to “1” and “2” (in a theoretical sense). Good course sequences on FEM spend a great deal of time on the implementation of these steps - in particular on step “3.” Recall that the expression given in step “3” was derived in Chapter 4.
The above “time-stepping algorithm” is just one possibility. The purpose here is merely to show one particular methodology that FEM software use for solving real-life problems. This methodology has only been shown to apply to isotropic elastic materials. Different time-stepping algorithms are surely used for other situations.

Plasticity

Obtaining Cauchy stress from the Jaumann rate is what is most often done by FEM software, even though the Truesdell rate form is better (exact) for elasticity. The Jaumann rate is known to have some problems with shear behavior due to the assumption of orthogonal bases [33]. In addition, for highly compressible materials, the Truesdell rate can give significantly better results [4]. Perhaps, then, the main reason that the Jaumann rate is preferred over the Truesdell rate (and other rates) has to do with plasticity.

In plasticity, many assumptions are made. The Truesdell rate turns out not to be good for plasticity, in general, because the dilatational (volume-changing) terms that relate \( \dot{\sigma} \) and the Truesdell rate are ignored in plasticity. So, suffice to say, plasticity is a complex topic that will not be covered in this introductory text, but the Jaumann rate, \( \dot{\sigma} \), is often used in FEM for solid elements, because it is computationally efficient and handles plasticity better than the Truesdell rate. A theoretical overview of implementation of plasticity in FEM can be found in [30].
Chapter 8

Linear Infinitesimal Elasticity

In hypoelasticity, we took the Cauchy stress, $\sigma$, and developed a constitutive relationship with the Left-Cauchy Green Strain Tensor, $\mathbf{B}$. This relationship was quite complicated (eq. (6.9)). Recall also that these tensors were both defined in spatial coordinates, by definition. Then, we derived an even more complicated “hypoelastic” formula (eq. (7.9)) that involved the rate of deformation tensor, $\mathbf{D}$. Lastly, a simple three-line pseudo-algorithm was given to illustrate how actual FEM (“solid” element) codes handle such “rate-form” constitutive expressions to solve, say, transient dynamic problems involving one or more bodies. We saw that this algorithm works equally nicely for linear infinitesimal elasticity as well.

In terms of both the non-rate form of the constitutive relationship and the rate-form, linear infinitesimal elasticity simplifies things considerably. Recall that the Second Piola Kirchhoff Stress Tensor, $\hat{\sigma}$, can be used for infinitesimal elasticity (review the example that starts on page 38). Since $\hat{\sigma}$ is defined in material coordinates, we will develop a constitutive relationship with a strain tensor that is similarly invariant to rigid body rotation, viz., $\mathbf{E}$.

Some authors instead choose to use the Cauchy stress and the Eulerian strain for the formulation of their linear infinitesimal constitutive expression. We are free to choose either stress-strain pair ($\sigma \leftrightarrow \mathbf{e}$ or $\hat{\sigma} \leftrightarrow \mathbf{E}$) in linear infinitesimal elasticity. We saw previously that our “time-stepping algorithm” was independent of this choice.

In this chapter we’ll take a look only at linear infinitesimal elasticity. If
a material is subjected to large strain (even if it is linear) or if a material is nonlinear (even if subjected to small strain), then it should be treated using the approaches developed in the previous chapters on hyperelasticity and hypoelasticity. Just as in our chapter on hyperelasticity, the focus here will be on developing a non-rate constitutive expression that relates stress and strain. In the elastic regime, a wide variety of materials, including most metals, behave linearly.

The theory of linear infinitesimal elasticity was developed independent of hyperelastic theory, with the former being mostly developed, originally, from intuition and experience [23]. Nevertheless, the linear theory can be derived as well, using the same starting assumptions that we used at in the previous hyperelasticity derivation.

Recall $\hat{\sigma} = \frac{d\phi}{dE}$

Infinitesimal elasticity $\rightarrow E \approx \epsilon, \ \hat{\sigma} \approx \sigma$

Note: The linear infinitesimal stress will be denoted by $\sigma$ for the remainder of this chapter. So long as $\epsilon$ and $\sigma$ are work conjugate, we can proceed in deriving our constitutive relationship.

We know the following three expressions to be true as well:

\[ \sigma = \frac{d\phi}{d\epsilon} \]
\[ \epsilon_{ij} = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \]
\[ x_i \approx X_i \text{ (still, } u_i = x_i - X_i) \]

In index notation, $\sigma_{ij} = \frac{d\phi}{d\epsilon_{ij}}$

For linear materials, we can define $\phi$, using a Taylor expansion, as follows:

\[ \phi(\epsilon_{ij}) = \phi(0) + \left( \frac{d\phi}{d\epsilon_{ij}} \right)_0 \epsilon_{ij} + \frac{1}{2} \left( \frac{d^2\phi}{d\epsilon_{ij}d\epsilon_{kl}} \right)_0 \epsilon_{ij} \epsilon_{kl} \]

Note: This is a taylor series with higher terms neglected

So, $\phi = a_0 + b_{ij} \epsilon_{ij} + \frac{1}{2} C_{ijkl} \epsilon_{ij} \epsilon_{kl}$

\[ \sigma_{ij} = \frac{d\phi}{d\epsilon_{ij}} = b_{ij} + \frac{1}{2} (C_{ijkl} + C_{klij}) \epsilon_{kl} \quad (8.1) \]
note: In eq. (8.1), \( \frac{d}{d\epsilon_{ij}} (C_{mnkl} \epsilon_{mn} \epsilon_{kl}) = C_{mnkl} (\delta_{mi} \delta_{nj} \epsilon_{kl} + \epsilon_{mn} \delta_{ik} \delta_{jl}) \)

\( = C_{ijkl} \epsilon_{kl} + C_{mnij} \epsilon_{mn} \) (or \( C_{klij} \epsilon_{kl} \))

If there is no initial stress:

\( b_{ij} = 0 \) (8.2)

Note that the order \( \frac{d^2 \phi}{d\epsilon_{ij} d\epsilon_{kl}} \) or \( \frac{d^2 \phi}{d\epsilon_{kl} d\epsilon_{ij}} \) doesn’t matter

\( C_{ijkl} = C_{klij} \) (8.3)

Eq. (8.3) \( \rightarrow \) Eq. (8.2) \( \rightarrow \) Eq. (8.1) \( \rightarrow \) \( \sigma_{ij} = C_{ijkl} \epsilon_{kl} \)

\( C_{ijkl} \) has \( 3^4 = 81 \) potential terms (9 x 9 matrix) to begin with.

Symmetry: \( \sigma_{ij} = \sigma_{ji} \) and \( \epsilon_{kl} = \epsilon_{lk} \) \( \rightarrow \) 36 terms (6 x 6 matrix)

Symmetry in \( C \rightarrow C_{ijkl} = C_{klij} \) (i.e. eq. (8.3) above)

\( \rightarrow \) 21 constants (general anisotropic)

A plane of symmetry is defined as follows: for every pair of coordinate systems that are mirror images of each other about the plane of symmetry, the elastic constants are the same. A single plane of symmetry reduces the number of independent constants to 13. Orthotropic symmetry (three orthogonal planes of symmetry) reduces the number of independent constants to 9. The proof can be found in [23].

Isotropic \( \rightarrow \) only 2 independent constants

Proof:
Recall \( \phi = \frac{1}{2} C_{ijkl} \epsilon_{ij} \epsilon_{kl} \) (valid outside of isotropy)

\begin{itemize}
  \item note: Since \( \phi \) is a function of strain only, path dependence is not considered (i.e. we are still only considering elasticity)
  \item Also note: \( \phi \) is quadratic
\end{itemize}

For isotropy: \( \phi = \phi(\epsilon) = \phi(I_\epsilon, II_\epsilon, III_\epsilon) \)
(recall that for hyperelastic isotropy, \( \phi = \phi(B) = \phi(I_B, II_B, III_B) \))

Let’s assume \( \phi = a I_\epsilon^2 + b II_\epsilon \), using the following reasoning:
CHAPTER 8. LINEAR INFINITESIMAL ELASTICITY

We know that \( \phi \) is quadratic. Thus, \( \phi \) doesn’t depend on \( III_\epsilon \), since \( III_\epsilon \) is cubic \( (\lambda_1\lambda_2\lambda_3) \). \( I_\epsilon \) is linear and so is, accordingly, squared, in order to obtain a quadratic. Our resulting \( I_\epsilon^2 \) term is also multiplied by a coefficient, \( a \). \( II_\epsilon = \lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_1\lambda_3 \) is, similarly, multiplied by a constant, \( b \).

Recall that \( I_\epsilon \sim (tr(\epsilon))^2 \) and \( I_\epsilon \) is a function of both \( (tr(\epsilon))^2 \) and \( tr(\epsilon^2) \).

Also recall that \( tr(\epsilon^2) = \epsilon_{ij}\epsilon_{ij} \), since \( \epsilon \) is a symmetric tensor.

\[
\frac{d\phi}{d\epsilon_{mn}} = a(\epsilon_{ii}\frac{d\epsilon_{jj}}{d\epsilon_{mn}} + \epsilon_{jj}\frac{d\epsilon_{ii}}{d\epsilon_{mn}}) + 2b[\delta_{im}\frac{d\epsilon_{jn}}{d\epsilon_{mn}} + \delta_{jn}\frac{d\epsilon_{im}}{d\epsilon_{mn}}] + 2b[\delta_{im}\delta_{jn}\epsilon_{mn}]
\]

* - note that “m” must equal “n” in the expression that is multiplied by “a”. In order to be non-zero.

So,

\[
\frac{d\phi}{d\epsilon_{mn}} = 2a\epsilon_{ii} + 2b\epsilon_{mn}
\]

\( 2a \rightarrow \lambda \)

\( b \rightarrow \mu \)

Recall \( \sigma_{ij} = \frac{d\phi}{d\epsilon_{ij}} : \)

\[
\sigma_{ij} = \lambda\epsilon_{kk}\delta_{ij} + 2\mu\epsilon_{ij}
\]

In eq. (8.4), \( \lambda \) and \( \mu \) are Lamé elastic constants. We can see that we have two independent constants, as expected. These constants are determined experimentally, as will be discussed shortly.

The following alternative expression is commonly found in literature on FEM:

\[
\sigma = \mathbf{C} : \epsilon
\]

We will see later what \( \mathbf{C} \) is, as well as why we use the “:” operator.
\[
\begin{bmatrix}
\sigma_{11} & \sigma_{12} & \sigma_{13} \\
\sigma_{21} & \sigma_{22} & \sigma_{23} \\
\sigma_{31} & \sigma_{32} & \sigma_{33}
\end{bmatrix} =
\begin{bmatrix}
\epsilon_{11} & \epsilon_{12} & \epsilon_{13} \\
\epsilon_{21} & \epsilon_{22} & \epsilon_{23} \\
\epsilon_{31} & \epsilon_{32} & \epsilon_{33}
\end{bmatrix}
\]

\[\epsilon^2\]

**note:** \(\mathbf{C}\) is symmetric in matrix form

Also note: The shear terms are multiplied by “2” because we used symmetry of \(\epsilon\) to reduce the 9 x 9 tensor, \(\mathbf{C}\), to a 6 x 6.

The matrix “\(C_{ijkl}\)” can be written in “Voigt” notation, “\(C_{IJ}\)” as well:

\[
\begin{bmatrix}
C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\
C_{21} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\
C_{31} & C_{32} & C_{33} & C_{34} & C_{35} & C_{36} \\
C_{41} & C_{42} & C_{43} & C_{44} & C_{45} & C_{46} \\
C_{51} & C_{52} & C_{53} & C_{54} & C_{55} & C_{56} \\
C_{61} & C_{62} & C_{63} & C_{64} & C_{65} & C_{66}
\end{bmatrix}
\]

Using Voigt notation, we can construct table 8.1:

<table>
<thead>
<tr>
<th>Table 8.1: Voigt Notation</th>
</tr>
</thead>
<tbody>
<tr>
<td>(I/J)</td>
</tr>
<tr>
<td>1</td>
</tr>
<tr>
<td>2</td>
</tr>
<tr>
<td>3</td>
</tr>
<tr>
<td>4</td>
</tr>
<tr>
<td>5</td>
</tr>
<tr>
<td>6</td>
</tr>
</tbody>
</table>

Table 8.1 can be simplified in this way due to the symmetries of the \(C_{ijkl}\) tensor apparent upon inspection.

*e.g.* the way you read the 1\(^{st}\) row depends on what information you need:

when \(I = 1\), \(i = 1\), \(j = 1\)
when \( J = 1, k = 1, l = 1 \)

This is easily confirmed by comparing, say, the “1-1” term in each of the above two matrices.

\[
C_{ijkl} = \lambda \delta_{ij} \delta_{kl} + \mu (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}) \quad (8.6)
\]

\( i.e. \)

\[
C_{45} = C_{2331} = \lambda \delta_{23} \delta_{31} + \mu (\delta_{22} \delta_{33} + \delta_{23} \delta_{32}) = 0
\]

\[
C_{44} = C_{2323} = \lambda \delta_{23} \delta_{23} + \mu (\delta_{22} \delta_{33} + \delta_{23} \delta_{32}) = \mu
\]

\textbf{note: Eq. (8.6) is how stiffness matrices are formed in FEM}

Sometimes we like to write the constitutive equation in the manner shown in eq. (8.5). We can prove that this expression is true by noting that the “:” operator is essentially two dot products (it can be easily shown that our previous definition of : for second order tensors still holds true as well).

\[
\sigma = [\lambda \delta_{ij} \delta_{kl} + \mu (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk})] e_i \otimes e_j \otimes e_k \otimes e_l : \epsilon_{mn} e_m \otimes e_n
\]

\( \text{can be thought of as a 9x9 matrix} \)

\[
= [\lambda \delta_{ij} \epsilon_{mn} + \mu (\delta_{im} \delta_{jn} + \delta_{in} \delta_{jm})] \epsilon_{mn} e_i \otimes e_j
\]

\[
= [\lambda \delta_{ij} \epsilon_{mm} + 2 \mu \epsilon_{ij}] e_i \otimes e_j \text{, which is as expected.}
\]

\textbf{note: symmetry of \( \epsilon_{mn} \) was invoked in the last step of the derivation}

### 8.1 Elastic Modulii and Poisson Ratio

What are the Lamé constants \( \mu \) and \( \lambda \)?

First, we’ll define the Bulk Modulus “\( k \)” by considering a block oriented along the principal strains:

Initially, \( V_0 = l_0 w_0 h_0 \)

After deformation, \( w = w_0 [1 + \lambda_1] \); \( l = l_0 [1 + \lambda_2] \); \( h = h_0 [1 + \lambda_3] \)
8.1. ELASTIC MODULII AND POISSON RATIO

\[ V = lwh \]

\[ \frac{V - V_0}{V} = \frac{\Delta V}{V_0} = \frac{l w h (1 + \lambda_1) l_0 (1 + \lambda_2) + (1 + \lambda_3) - l_0 w h_0}{l_0 w h_0} = (1 + \lambda_1)(1 + \lambda_2)(1 + \lambda_3) - 1 \]

\[ \lambda_1 + \lambda_2 + \lambda_3 + O(\lambda^2) \approx \varepsilon_{kk} \]

Now, from eq. (8.4), we know that \( \sigma_{ii} \) can be expressed as follows:

\[ \sigma_{ii} = \lambda \delta_{ii} \varepsilon_{kk} + 2\mu \varepsilon_{ii} \]

Summing both sides on \( i \) (and on \( k \), as always) and noting that \( \sum \frac{\sigma_{ii}}{3} = -P \)

(pressure - e.x. hydrostatic)

\[ \rightarrow \sum \frac{\sigma_{ii}}{3} = \left( \lambda + \frac{2}{3}\mu \right) \varepsilon_{kk} \]

or

\[ P = -k \Delta V, \text{ where } k = \text{“bulk modulus”} \]

note: \( \frac{1}{k} \) is sometimes called the “compressibility”, since “\( k \)” relates hydrostatic pressure to volumetric strain. Really though, compressibility is determined from \( \frac{k}{G} \), where “\( G \)” is the “shear modulus”. \( \frac{k}{G} \gg 1 (\rightarrow \infty) \rightarrow \nu \rightarrow \frac{1}{2} \), where “\( \nu \)” is the Poisson Ratio. “\( \nu \)” = \( \frac{1}{2} \) \( \rightarrow \) incompressible material.

Next, we’ll define the Young’s Modulus, “\( E \)”, the Poisson Ratio, “\( \nu \)”, and the shear modulus, “\( G \)”. Any two modulii are needed to define an isotropic material \( (E, \nu \text{ or } k, G, \text{ etc.}) \)

We need \( \varepsilon \) in terms of \( \sigma \);

We’ll start with eq. (8.4), which is re-written, below:

\[ \sigma_{ij} = \lambda \varepsilon_{kk} \delta_{ij} + 2\mu \varepsilon_{ij} \quad (1) \]

Take the trace:

\[ \sigma_{11} = \lambda (\varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33}) + 2\mu \varepsilon_{11} \]

\[ \sigma_{22} = \lambda (\varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33}) + 2\mu \varepsilon_{22} \]

\[ \sigma_{33} = \lambda (\varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33}) + 2\mu \varepsilon_{33} \]

\[ \sigma_{11} + \sigma_{22} + \sigma_{33} = 3\lambda \varepsilon_{11} + 3\lambda \varepsilon_{22} + 3\lambda \varepsilon_{33} + 2\mu \varepsilon_{11} + 2\mu \varepsilon_{22} + 2\mu \varepsilon_{33} \]
\( \rightarrow \sigma_{kk} = 3\lambda \epsilon_{kk} + 2\mu \epsilon_{kk} = (3\lambda + 2\mu) \epsilon_{kk} \)

\( \rightarrow \epsilon_{kk} = \frac{\sigma_{kk}}{3\lambda + 2\mu} \quad (2) \)

(2) \(\rightarrow\) (1) and invert:

\( \epsilon_{ij} = \frac{1}{2\mu} (\sigma_{ij} - \frac{\lambda}{3\lambda + 2\mu} \sigma_{kk} \delta_{ij}) \quad (8.7) \)

Now we’re ready to find \( E, \nu, G \):

Consider the case of uniaxial tension: \( \sigma = \begin{bmatrix} \sigma_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \)

\( \epsilon_{11} = \frac{\sigma_{11}}{2\mu} (1 - \frac{\lambda}{3\lambda + 2\mu}) = \frac{2(\lambda + \mu)}{2\mu(3\lambda + 2\mu)} \sigma_{11} \)

\[ \star - \text{must be } \frac{1}{E} \text{ since those of us that have done laboratory testing of linear, isotropic, materials, know that } \sigma = E\epsilon \text{ for a simple uniaxial test} \]

\( E = \frac{\mu(3\lambda + 2\mu)}{\lambda + \mu} \quad (8.8) \)

Now, \( \epsilon_{22} = \frac{1}{2\mu} \left( \sigma_{22} - \frac{\lambda}{3\lambda + 2\mu} (\sigma_{11} + \sigma_{22} + \sigma_{33}) \right) = -\frac{\lambda}{2\mu(3\lambda + 2\mu)} \sigma_{11} \)

where \( \epsilon_{22} \) is the lateral strain that occurs from the uniaxial (longitudinal) stress.

By definition (\( i.e. \) as defined in undergraduate mechanics of materials),

\( \nu = -\frac{\epsilon_{22}}{\epsilon_{11}} = \frac{\frac{\lambda}{2(\lambda + \mu)} \sigma_{12}}{\frac{\lambda}{2(\lambda + \mu)} \sigma_{11}} = \frac{\lambda}{2(\lambda + \mu)} \quad 0 < \nu < 1/2 \quad (8.9) \)

Lastly, consider pure shear: \( \sigma = \begin{bmatrix} 0 & \sigma_{12} & 0 \\ \sigma_{12} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \)
8.2. NAVIER AND BELTRAMI-MICHELL EQUATIONS

\[ \sigma_{12} = \lambda \epsilon_{kk} \delta_{12} + 2 \mu \epsilon_{12} = 2 \mu \epsilon_{12} \]

Since \( \gamma = 2 \epsilon_{12} \) and \( \sigma_{12} = G \gamma \) (by definition),

\[ G = \mu \quad (8.10) \]

It can also be easily shown that \( G = \frac{E}{2(1+\nu)} \)

8.2 Navier and Beltrami-Michell Equations

Summary of equations for infinitesimal linear isotropic elasticity (static equilibrium only):

There are 3 equilibrium equations (6 unknowns):

\[ \frac{\partial \sigma_{ij}}{\partial x_j} + \rho b_i = 0 \quad \text{(no sum on } i) \quad (8.11) \]

There are 6 constitutive equations (6 more unknowns):

\[ \sigma_{ij} = 2 \mu \epsilon_{ij} + \lambda \epsilon_{kk} \delta_{ij} \quad (8.12) \]

There are 6 strain-displacement equations (3 more unknowns - i.e. displacement in 3 directions):

\[ \epsilon_{ij} = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \quad (8.13) \]

TOTAL = 15 equations ; 15 unknowns

By this point, we have dealt with all of these equations enough to recognize that there are indeed 15 unknowns and 15 equations. However, in order to stem any possible confusion about the physical interpretation of these equations and unknowns, it will be mentioned one last time that rotational values of “\( u \)” as well as “moment” equilibrium of stress only appear in structural analysis formulas that have been simplified. In solid mechanics, concepts
like “moment equilibrium” are not considered, but would be satisfied naturally in any well-posed problem.

\[ \text{eq. (8.13)} \longrightarrow \text{eq. (8.12)} \longrightarrow \text{eq. (8.11)} \]

\[ \longrightarrow 3 \text{ equations, 3 unknowns (3 displacements):} \]

\[ (\lambda + \mu) \frac{\partial^2 u_j}{\partial x_i \partial x_j} + \mu \frac{\partial^2 u_i}{\partial x_j \partial x_j} + b_i = 0 \quad i = 1, 2, 3 \quad (8.14) \]

The three equations of eq. (8.14) are known as the Navier Equations.

Just as the Navier Equations (eq. (8.14)) depict a “displacement” relationship, we can alternatively develop a “stress” relationship, known as the Beltrami-Michell Equations.

Without going through the derivation:

\[ \sigma_{ij, kk} + \frac{1}{1 + \nu} \sigma_{kk, ij} - \frac{\nu}{1 + \nu} \sigma_{mm, kk} \delta_{ij} = 0 \quad (8.15) \]

The six equations of eq. (8.15) are known as the Beltrami-Michell Compatibility Equations. In 2D, we can derive a similar expression with the aid of the so-called Airy Stress Function. Energy methods can also be used in order to arrive at similarly concise expressions. Graduate-level courses in solid mechanics devoted to linear infinitesimal elasticity would investigate these kinds of concepts in detail.
8.3 Final Words

This text is intended to be a broad introduction to isotropic elasticity, with a focus on providing complete derivations for strain, stress, and their constitutive relationships for both “finite” hyperelasticity and “infinitesimal” linear elasticity. The complications that arise due to rigid body rotations were discussed, and the importance of “work-conjugate” tensor pairs was emphasized. A particular “rate” approach to the handling of rigid body rotations was discussed, which included a particular treatment for the “frame of reference” problem with respect to our hyperelasticity model as well as for our linear infinitesimal elasticity model.

Specific hyperelastic material models were introduced, along with a description of specific methodologies that are currently used for fitting to experimental data. Lastly, the constitutive equations for linear infinitesimal elasticity were developed, including the derivation of Young’s Modulus and Poisson Ratio, which are material properties that may be familiar to the reader.

We concluded this chapter by considering all of the important equations in solid mechanics: the equations of equilibrium, the equations relating stress and strain, and the equations relating strain to displacement. These, or similar, equations, are used in any FEM code on the market. With the completion of this text, we now have the foundation that is necessary in order to investigate many challenging engineering and materials problems. We also have the tools to study more specific topics in solid mechanics, such as anisotropy, viscoelasticity, and metal plasticity. Further research in the area of solid mechanics is also needed, particularly in the treatment of brittle-type damage, for those so inclined.
Appendix A

Math derivations

A.1 Transpose of tensor product

We'd like to prove that $A^T = C^T \cdot B^T$, if $A$ is defined as $A = B \cdot C$.

In other words, we will show that $(B \cdot C)^T = C^T \cdot B^T$

We can start with a particular definition of transpose, namely,

$$b \cdot A^T \cdot a = a \cdot A \cdot b$$ \hspace{1cm} (A.1)

In eq. (A.1), vectors $a$ and $b$ are arbitrary.

Using this definition of transpose (eq. (A.1)), and then the associative law for vector and tensor dot products,

$$a \cdot (B \cdot C)^T \cdot b = b \cdot (B \cdot C) \cdot a = b \cdot B \cdot (C \cdot a)$$

By invoking the definition of transpose, and then the commutative law of addition for vector dot products, and then the associative law for products,

$$a \cdot (B \cdot C)^T \cdot b = (C \cdot a) \cdot B^T \cdot b = B^T \cdot b \cdot (C \cdot a) = (B^T \cdot b) \cdot C \cdot a$$

Now, we can once again invoke the definition of transpose, and then the associative law for vector and tensor dot products:

$$a \cdot (B \cdot C)^T \cdot b = a \cdot C^T \cdot (B^T \cdot b) = a \cdot (C^T \cdot B^T) \cdot b$$
Therefore, $(\mathbf{B} \cdot \mathbf{C})^T = \mathbf{C}^T \cdot \mathbf{B}^T$
A.2 Skew tensor

In order to show that $W$ is “skew,” we need to show that $W = -W^T$. It is sufficient to show that $L - L^T = -(L - L^T)^T$.

Similar to the proof shown in Appendix A.1, we will start with arbitrary vectors $a$ and $b$.

Consider that the definition of transpose (eq. (A.1)) states that:

$$a \cdot (L - L^T)^T \cdot b = b \cdot (L - L^T) \cdot a$$

We can expand the last term and again invoke the definition of transpose:

$$a \cdot (L - L^T)^T \cdot b = b \cdot L \cdot a - b \cdot L^T \cdot a = a \cdot L^T \cdot b - a \cdot L \cdot b$$

Finally, we see that the last term can be reduced:

$$a \cdot (L - L^T)^T \cdot b = a \cdot (L^T - L) \cdot b = a \cdot -(L - L^T) \cdot b$$

This proves that $W$ is indeed anti-symmetric or “skew.”
A.3 Orthogonal tensor

Orthogonality of $\mathbf{R}$ will be proven if we can show that the product of $\mathbf{R}$ with its transpose yields the identity tensor, $\mathbf{I}$.

Since $\mathbf{F} = \mathbf{R} \cdot \mathbf{U}$, we know that $\mathbf{R} = \mathbf{F} \cdot \mathbf{U}^{-1}$

So, $\mathbf{R}^T \cdot \mathbf{R} = (\mathbf{F} \cdot \mathbf{U}^{-1})^T \cdot (\mathbf{R} \cdot \mathbf{U}^{-1})$

From Appendix A.1, we know that $(\mathbf{F} \cdot \mathbf{U}^{-1})^T = (\mathbf{U}^{-1})^T \cdot \mathbf{F}^T$

Thus,

$$\mathbf{R}^T \cdot \mathbf{R} = (\mathbf{U}^{-1})^T \cdot \mathbf{F}^T \cdot \mathbf{F} \cdot \mathbf{U}^{-1} = \mathbf{U}^{-1} \cdot \mathbf{F}^T \cdot \mathbf{F} \cdot \mathbf{U}^{-1} = \mathbf{U}^{-1} \cdot \mathbf{U}^2 \cdot \mathbf{U}^{-1}$$

Since $\mathbf{U}^{-1} \cdot \mathbf{U}^2 \cdot \mathbf{U}^{-1}$ reduces to $\mathbf{I}$, we have the desired result.
Appendix B

Stress derivations

B.1 Physical interpretation of $\sigma$

Consider the stress "wedge," or "Cauchy tetrahedron," depicted in Fig. B.1, where we would like to know the stresses acting on the plane defined by a unit vector $\mathbf{n}$:

$\mathbf{n} = n_i \mathbf{e}_i$

Noting the triangle $AOD$ in Fig. B.1, with angle $\alpha_1$:

$\cos \alpha_1 = \frac{dh}{dx_1}$

We note here that $\cos \alpha_1$ is the component of $\mathbf{n}$ in the "1" direction.

So, $dx_1 = \frac{dh}{n_1}$

Now, $dV = \frac{V}{\beta} dS \cdot dh = \frac{V}{\beta} dx_1 dS_1 = \frac{V}{\beta} dx_2 dS_2 = \frac{V}{\beta} dx_3 dS_3$
APPENDIX B. STRESS DERIVATIONS

The above expression for $dV$ may be hard to visualize, but we’re essentially multiplying a plane by a distance, analogous to $V = Ah$ for a cylinder.

Solving for, say, $dS_1 \rightarrow dS_1 = \frac{dS dh}{dx_i} \ldots$ etc..

In general, $dS_i = dS \frac{dh}{dx_i}$

We note also that this above expression can be re-written, since $n_i = \frac{dh}{dx_i}$:

$$dS_i = dSn_i$$

(B.1)

i.e. If $n_1$ is small, then the surface $dS_1$ defined by $n$ and shown in Fig. B.1, is small ($dx_1$ is large).

![Figure B.2: Stresses](image)

If $\rho b dV$ is the “body force” (force due to gravity, for example), and $t$ is stress (Fig. B.2), then:

$$\sum F : t_n dS - \sum t_i dS_i + \rho b dV = \rho dV \frac{dv}{dt},$$

where $\rho dV \frac{dv}{dt}$ is essentially mass * acceleration

Take $dh \rightarrow 0$ since we want the stresses at a point:
Divide through by \( dS \) and note that \( \frac{dV}{dS} \to 0 \) as \( dh \to 0 \), \( \mathbf{t_n} = \sum t_i \frac{dS_i}{dS} \)

But, it was previously shown (eq. (B.1)) that \( \frac{dS_i}{dS} = n_i \).

So, \( \mathbf{t_n} = \sum t_i n_i \), where \( t_i = n_i t_i \), as depicted in Fig. B.2.

Consider, for example, \( \mathbf{t_2} \), shown in Fig. B.3:

\[
\text{note: Normal and shear stress magnitudes are commonly denoted by } \sigma.
\]

So, \( \mathbf{t_2} = [\sigma_{21}, \sigma_{22}, \sigma_{23}] \)

Here, the first subscript may be thought of as face “2” (Fig. B.2) and the second subscript can be thought of as the direction of stress on that particular face.

\( \mathbf{t_2} = \sigma_{2i} \mathbf{e_i} \)

In general, \( \mathbf{t_i} = \sigma_{ij} \mathbf{e_j} \)

We know that \( \mathbf{t_n} = \sum t_i n_i \)

Substituting, we get \( \mathbf{t_n} = \sum \sigma_{ij} \mathbf{e_j} n_i = \sigma_{ij} \mathbf{e_j} n_i \) (index notation)

If \( \mathbf{n} = n_i \mathbf{e_i} \) and \( \mathbf{e_j} \), then \( \mathbf{t_n} = \sigma_{ij} \mathbf{e_j} n_i \) is matrix-vector multiplication so long as \( \sigma \) is a symmetric tensor (recall eq. (1.9) for the expression that defines matrix-vector multiplication, regardless of symmetry).
\[ t_n = \sigma \cdot n \tag{B.2} \]

Since we are really concerned with points in a body rather than volumes (recall that we took \( dh \rightarrow 0 \) earlier), the physical meaning of eq. (B.2) is essentially as follows: if we know the normal and shear stresses at a particular point in a body (with respect to a particular bases), then we can find the stresses in any direction (or at any angle).

\[ \sigma = \sigma^T \tag{B.3} \]

The most common proof of eq. (B.3) involves summing moments (\( i.e. \) “conservation of angular momentum”) and since we skipped how to do cross products, we’ll skip this proof (the complete proof can be found in [2]).
B.2. EQUATION OF MOTION

B.2 Equation of Motion

Let’s start with:

\[ \int_S \sigma \cdot n \, dS + \int_V \rho b \, dV = \frac{d}{dt} \int_V \rho v \, dV \quad (B.4) \]

Our goal is to arrive at the following result:

\[ \int_V \frac{\partial \sigma_{ij}}{\partial x_j} \, dV + \int_V \rho b_i \, dV = \int_V \rho \frac{dv_i}{dt} \, dV \quad (B.5) \]

In order to accomplish this, we must make several observations. First, we need the divergence theorem for second-order tensors.

For vectors, the divergence theorem can be written:

\[ \int_S u \cdot n \, dS = \int_V \text{div}(u) \, dV \quad (B.6) \]

In eq. (B.6), \( \text{div}(u) = \frac{\partial u_i}{\partial x_i} \)

For tensors, we have:

\[ \int_S \sigma \cdot n \, dS = \int_V \text{div}(\sigma) \, dV \quad (B.7) \]

In eq. (B.7), \( \text{div}(\sigma) = \frac{\partial \sigma_{ij}}{\partial x_j} \)
We can derive eq. (B.7) as follows:

Let \( \mathbf{b} \) be an arbitrary vector:

\[
\mathbf{b} \cdot \int_S \mathbf{\sigma} \cdot \mathbf{n} \, dS = \int_S \mathbf{b} \cdot \mathbf{\sigma} \cdot \mathbf{n} \, dS
\]

Using the definition of transpose (Appendix A.1) and the associative law for vectors, we get:

\[
\int_S (\mathbf{\sigma}^T \cdot \mathbf{b}) \cdot \mathbf{n} \, dS
\]

From the divergence theorem for vectors (eq. (B.6)), we get:

\[
\int_S (\mathbf{\sigma}^T \cdot \mathbf{b}) \cdot \mathbf{n} \, dS = \int_V \text{div}(\mathbf{\sigma}^T \cdot \mathbf{b}) \, dV
\]

Note that:

\[
\text{div}(\mathbf{\sigma}^T \cdot \mathbf{b}) = \frac{\partial}{\partial x_i} (\mathbf{\sigma}^T \cdot \mathbf{b}) = \sigma_{ki} \frac{\partial b_k}{\partial x_i} + \sigma_{ki} \frac{\partial b_k}{\partial x_i} = \text{div}(\mathbf{\sigma}^T) \cdot \mathbf{b}
\]

where the slashed term is zero.

To complete the derivation, note that \( \mathbf{\sigma} = \mathbf{\sigma}^T \) from Appendix B.1. Additionally, noting the associative law of vectors, we get the desired result, namely:

\[
\mathbf{b} \cdot \int_S \mathbf{\sigma} \cdot \mathbf{n} \, dS = \int_V \mathbf{b} \cdot \text{div}(\mathbf{\sigma}) \, dV = \mathbf{b} \cdot \int_V \text{div}(\mathbf{\sigma}) \, dV
\]

The first term in eq. (B.7) is now understood, but to derive the last term in eq. (B.7), we need to make two additional observations.

First, we know from eq. (2.9) that:

\[
dV = \det(\mathbf{F}) \, dV_0 \tag{B.8}
\]

We can similarly express \( \rho \) in terms of \( \rho_0 \) by noting from the conservation of mass, that \( \int_V \rho \, dV = \int_{V_0} \rho_0 \, dV_0 \)

Substituting eq. (B.8), we get \( \int_V \rho \, dV = \int_{V_0} \rho \det(\mathbf{F}) \, dV_0 = \int_{V_0} \rho_0 \, dV_0 \)

Thus, we have \( \int_{V_0} (\rho \det(\mathbf{F}) - \rho_0) \, dV_0 = 0 \), which is true for any arbitrary \( V_0 \). So,

\[
\rho = \frac{\rho_0}{\det(\mathbf{F})} \tag{B.9}
\]

Finally, substituting eq. (B.8) and eq. (B.9) into the last term in eq. (B.4), we get:

\[
\frac{d}{dt} \int_V \rho \, v \, dV = \int_{V_0} \frac{d}{dt} \left( \frac{\rho_0}{\det(\mathbf{F})} \right) \, \nu \det(\mathbf{F}) \, dV_0
\]

Since \( \rho_0 \) and \( V_0 \) are constant with time, we have:
The final step is to substitute $\rho_0 = \rho \det F$ and $dV_0 = \frac{dV}{\det(F)}$ into eq. (B.10). We then arrive at the expected result:

$$\frac{d}{dt} \int_V \rho \nu dV = \int_{V_0} \rho_0 \frac{d\nu}{dt} dV_0$$  \hspace{1cm} (B.10)

Substituting eq. (B.7) and eq. (B.11) into eq. (B.4), we get the desired result (eq. (B.5)).
Appendix C

Hyperelastic derivations

C.1 Proof of $\sigma = f(B)$

Assume $\sigma$ is a function of $F$

We know from the Chapter 5, that for a superimposed strain:

$F^* = Q \cdot F$

$\sigma^* = Q \cdot \sigma \cdot Q^T$

So, $Q \cdot \sigma \cdot Q^T = f(Q \cdot F)$

From polar decomposition, we know $F = V \cdot R = R \cdot U$

We will first consider the latter form of $F$. Let’s take $Q = R^T$, where $Q$ can be any orthogonal tensor. To see why we pick $Q = R^T$, recall that $U$ is defined in “material” coordinates, and so $U$ is, accordingly, invariant to any rigid body rotation. $\sigma$ is defined in spatial coordinates and is not invariant to rotation. So, naturally, $\sigma = f(F) = f(R \cdot U)$ is a function of both $U$ and $R$, rather than just $U$. We can take $\sigma^* = f(F^*)$ and set $Q$ equal to $R^T$, for convenience.

$R^T \cdot \sigma \cdot R = f(R^T \cdot R \cdot U) \rightarrow \sigma = R \cdot f(R^T \cdot R \cdot U) \cdot R^T$

$\sigma = R \cdot f(U) \cdot R^T$

Recall that $C = U^2$

$\sigma = R \cdot g(C) \cdot R^T$  \hspace{1cm} (C.1)
Perhaps eq. (C.1) seems obvious, since $\sigma^* = Q \cdot \sigma \cdot Q^T$ and $C^* = C$, but we started with $\sigma = f(F)$ for completeness.

Now, let's apply $Q$ before deformation:

![Figure C.1: Rigid body rotation applied prior to deformation](image)

We can define $dX^*$ as follows:

$$dX^* = Q \cdot dX$$

Recall that last time we superimposed a rigid body rotation on $dx$, which resulted in $dx^* = Q \cdot dx$.

This time, we want $dx^* = dx$

We can see from Fig. C.2, that we need $F^*$ to be a function of $Q^T$

We can show that this is indeed the case as follows:

$$dx^* = dx = F^* \cdot Q \cdot dX$$

Since $dx = F \cdot dX$, we find that $F = F^* \cdot Q$, which yields:
C.1. PROOF OF $\sigma = F(B)$

Figure C.2: Rigid body rotation applied prior to deformation

"F" only stretches in this case, for the purposes of illustration

(i.e. if $F^* = F$ and $dx^* = F^* \cdot dX^*$, then we won't get $dx^* = dx$)
\[ F^* = F \cdot Q^T \]  \hspace{1cm} (C.2)

Eq. (C.2) is what we wanted to find, and it should be expected. Recall that \( V \cdot R \) is physically understood to be a rotation, \( R \), followed by a deformation (axial strains and shear), \( V \). Thus, \( F^* = F \cdot Q^T \) is an explicit rigid body rotation, \( Q^T \), followed by the total deformation + rotation, \( F \). In other words, the rotation, \( Q^T \), is applied to the initial configuration \( dX \).

Our new definition of \( F^* \) is difficult to physically interpret from a Lagrangian point-of-view, but we will use it in order to show that \( \sigma = f(V) \), as follows.

We know that \( \sigma^* = f(F^*) \), where our new definition of “*” requires that \( \sigma^* = \sigma \) and \( F^* = F \cdot Q^T \).

So, \( \sigma = f(F \cdot Q^T) \)

We know that \( F = V \cdot R \)

Substituting \( \sigma = f(V \cdot R \cdot Q^T) \)

Again, since we are simply applying our “*” operator to both \( \sigma \) and \( f(F) \), we can take \( Q \) to be anything we want, as doing so is analogous to operating on both sides of any ordinary equation. Here, we can take \( Q \) to be \( R \).

This allows us to directly arrive at the desired result:

\[ \sigma = f(V) \]  \hspace{1cm} (C.3)

The result (eq. (C.3)) is expected since it was shown in Chapter 5 that \( \sigma \) and \( V \) are work-conjugate. Recall also that \( V^2 = B \).
C.2 Derivation: \( \frac{dI_B}{dB}, \frac{dII_B}{dB}, \frac{dIII_B}{dB} \)

Since \( I_B = \text{tr}B = B_{kk} \), \( \frac{dI_B}{dB} = \frac{\partial B_{mn}}{\partial B_{kl}} e_k e_l = \delta_{nk} \delta_{ml} e_k e_l \) \( (C.4) \)

\[
\frac{dII_B}{dB} = \frac{1}{2} \left[ \frac{d\left(\text{tr}(B)^2 - \text{tr}(B^2)\right)}{dB} \right] = \frac{1}{2} \left[ 2 \text{tr}(B) \frac{d(B)}{dB} \right] - \frac{d(\text{tr}(B^2))}{dB} \\
= \frac{1}{2} \left[ \text{tr}(B) I - \frac{d(\text{tr}(B^2))}{dB} \right] \text{ chain rule} \\
\frac{d(\text{tr}(B^2))}{dB} = \frac{d(B_{ij}B_{ji})}{dB} = \frac{\partial B_{ij}}{\partial B_{mn}} B_{mn} e_n e_m = \frac{\partial B_{ij}}{\partial B_{mn}} B_{mn} + \frac{\partial B_{ji}}{\partial B_{mn}} B_{mn} B_{ij} e_n e_m \\
= \delta_{im} \delta_{jn} B_{ji} e_n e_m + \delta_{jm} \delta_{in} B_{ij} e_n e_m = B_{ji} e_j e_l + B_{ij} e_i e_j = 2 B_{ij} e_i e_j \\
\text{So, } \frac{dII_B}{dB} = \frac{1}{2} \left[ 2 \text{tr}(B) I - 2B_{ij} e_i e_j \right] = I_B I - B \quad (C.5) \]
We know $III_B = det B$; but we need a better expression for $III_B$ before we derive $\frac{dIII_B}{dB}$.

We know from eq. (1.15): $B^3 - I_B B^2 + II_B B - III_B I = 0$;

$tr(B^3) - I_B B^2 + II_B B - III_B I = tr(0)$

$tr(B^3) - tr(I_B B^2) + tr(II_B B) - tr(III_B I) = 0$

$\frac{I_B tr(B^2)}{II_B trB} \to I_B * III_B^* 3$

$tr(B^3) - tr(B)tr(B^2) + \frac{1}{2}[(tr(B^2)^2 - tr(B^2))]trB = I_B * 3$

$tr(B^3) - tr(B)tr(B^2) + \frac{1}{2} tr(B)tr(B)^2 - \frac{1}{2} tr(B)tr(B^2) = I_B * 3$

\[
\frac{1}{3} \left[ tr(B^3) - \frac{2}{3} tr(B^2)tr(B^2) + \frac{1}{6} (tr(B^3)^3) \right] = III_B
\]

\[
\frac{dIII_B}{dB} = \frac{d(1/3tr(B^3))}{dB} - \frac{d(1/2tr(B^2))}{dB} + \frac{d(1/6(tr(B^3)^3))}{dB}
\]

\[
= \frac{1}{3} \frac{d(tr(B^3))}{dB} - \frac{1}{2} tr(B^2) \frac{d(trB)}{dB} - \frac{1}{2} tr(B) \frac{d(tr(B^2))}{dB} - \frac{1}{2} tr(B) \frac{d(tr(B^2))}{dB}
\]

\[\text{product rule}\]

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\[\text{product rule}\]

\[\text{product rule}\]

where $\frac{d}{dB}(B^3) = \delta[B_{kl}B_{lm}B_{mk}]e_k e_l e_j$

$\frac{\partial B_{jk}}{\partial B_{ij}} B_{lm}B_{mk}e_k e_j + B_{kl} \frac{\partial B_{km}}{\partial B_{ji}} B_{mk}e_k e_j + B_{kl}B_{lm} \frac{\partial B_{mk}}{\partial B_{ji}} e_k e_j$

$\delta_{kj} \delta_{li} B_{lm} e_k e_j + \delta_{lj} \delta_{mi} B_{kl} e_k e_j + \delta_{li} \delta_{mj} B_{kl} e_k e_j + \delta_{mj} \delta_{ki} B_{kl} e_k e_j$

$B_{lm} B_{mj} e_k e_j + B_{kj} B_{lm} e_k e_j + B_{il} B_{lj} e_k e_j = 3B^2$

So,

$\frac{dIII_B}{dB} = \frac{1}{3}(3B^2) - \frac{1}{2} tr(B^2) I - \frac{1}{2} I_B (2B) + 1/6 (tr(B^2))^2 I$

$= B^2 - I_B B - 1/2 (tr(B^2) - I_B^2) I$

$= B^2 - I_B B + II_B I$, Since $II_B = \frac{1}{2} (I_B^2 - tr(B^2))$ (C.6)
C.2. DERIVATION: \( \frac{DB}{DB}, \frac{DB}{DB}, \frac{DB}{DB} \)
C.3 Principal stretch constitutive relationship

Recall from an earlier chapter (example problem at the end of the section on Polar Decomposition) that we can form either the Right Stretch Tensor, \( U \), or the Left Stretch Tensor, \( V \), in their principal stress space by pre and post multiplying by the orthogonal tensor, \( \Phi \), where \( \Phi \) contains either the eigenvectors of the \( U \) or the eigenvectors of \( V \), as appropriate. Also recall that the eigenvalues of either tensor are the same, and their invariants are the same.

\[
[U]_n = [\Phi]^T [U] [\Phi] 
\]

where \( [\Phi] = [\Phi]_U = \begin{bmatrix} (n_1) \lambda_1 & (n_1) \lambda_2 & (n_1) \lambda_3 \\ (n_2) \lambda_1 & (n_2) \lambda_2 & (n_2) \lambda_3 \\ (n_3) \lambda_1 & (n_3) \lambda_2 & (n_3) \lambda_3 \end{bmatrix} \)

\[
[\Phi]^T [U] [\Phi] = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} 
\]

\[
[\Phi]^T [E] [\Phi] = \frac{1}{2} \begin{bmatrix} \lambda_1^2 - 1 & 0 & 0 \\ 0 & \lambda_2^2 - 1 & 0 \\ 0 & 0 & \lambda_3^2 - 1 \end{bmatrix} = \frac{1}{2} (\lambda^2 - 1) 
\]

\[
[\Phi]^T [\hat{\sigma}] [\Phi] = \frac{\partial \phi}{\partial \Phi} (\text{where } \hat{\sigma} = \frac{\partial \phi}{\partial E} = 2 \frac{\partial \phi}{\partial C} = 2 \frac{\partial \phi}{\partial U^2}) 
\]

We know that the relationship between \( \sigma \) and \( \hat{\sigma} \) is \( \sigma = \frac{1}{III_U} F \cdot \hat{\sigma} \cdot F^T \).

Applying the operator \( \Phi^T \cdot A \cdot \Phi \) to all tensors in the above, we get:

\[
\Phi^T \cdot \sigma \cdot \Phi = \frac{1}{III_U} \Phi^T \cdot R \cdot U \cdot \Phi \frac{\partial \phi}{\partial \Phi^T \cdot E \cdot \Phi} \Phi^T \cdot U^T \cdot R^T \cdot \Phi 
\]

Pre-multiply \( \Phi^T \cdot R^T \cdot \Phi \) and post-multiply \( \Phi^T \cdot R \cdot \Phi \) to the above, and replace \( \frac{\partial \phi}{\partial \Phi^T \cdot E \cdot \Phi} \) with \( 2 \frac{\partial \phi}{\partial \Phi^T \cdot U^T \cdot \Phi} \):

\[
\Phi^T \cdot R^T \cdot \sigma \cdot R \cdot \Phi = 2 \frac{\partial \phi}{\partial \Phi^T \cdot U^T \cdot \Phi} \Phi^T \cdot U^T \cdot \Phi 
\]

We now note that \( U \) is symmetric, and \( III_U \) (which is the same as \( III_V = III_F = III_B^{1/2} \)) can be expressed in terms of the principal stretches, as \( \lambda_1 \lambda_2 \lambda_3 \) (recall chapter 1). Furthermore, we can note that \( \sigma^* = R^T \cdot \sigma \cdot R \) from Appendix C.1, for example, where we took “\( Q \)” to be equal to \( R^T \).

We recall that \( \sigma^* \) was indeed found to be equal to some function \( f(U) \).

\[
\Phi^T \cdot R^T \cdot \sigma \cdot R \cdot \Phi = \frac{2}{\lambda_1 \lambda_2 \lambda_3} \Phi^T \cdot U \cdot \Phi \frac{\partial \phi}{\partial \Phi^T \cdot U^T \cdot \Phi} \Phi^T \cdot U \cdot \Phi \quad (C.7) 
\]
In order to arrive at our desired result, which expresses the principal values of the Cauchy stress, $\sigma_i$, as a function of the principal stretches, we’d like to pre multiply by $R$ and post-multiplying by $R^T$. This would give us the desired result on the left-hand-side.

Since $R \cdot U = V \cdot R$, we can see that $R \cdot U \cdot R^T = V$

In addition, $U^2$ is $F^T \cdot F \rightarrow R \cdot U^2 \cdot R^T = R \cdot R^T \cdot V^2 \cdot R^T = V^2$

So, pre multiplying by $R$ and post-multiplying by $R^T$ on eq. (C.7), yields:

$$\Phi^T \sigma \Phi = \frac{2}{\lambda_1 \lambda_2 \lambda_3} \Phi^T \cdot V \cdot \Phi \frac{\partial \phi}{\partial \Phi^T} \cdot V^2 \cdot \Phi$$  \hspace{1cm} (C.8)

To arrive at our final result, we need to make a few more observations. Namely, we observe that all $\Phi$ that are present in eq. (C.8) and that may have been up to this point assumed to represent $\Phi_U$ can easily be replaced by $\Phi_V$ without affecting any of the algebra that we’ve already done. $\Phi$ can be any orthogonal tensor at this point. In fact, we’ve seen eq. (C.8) before, but without the presence of $\Phi$ (i.e. $\Phi = I$ yields eq. (6.2)). By taking $\Phi = \Phi_V$, we can now replace $\Phi^T \cdot V \cdot \Phi$ with $\lambda$ and $\Phi^T \cdot V^2 \cdot \Phi$ with $\lambda^2$.

In addition, we note that $\frac{\partial \phi}{\partial \lambda} = \frac{\partial \phi}{\partial f(\lambda)} \frac{\partial f(\lambda)}{\partial \lambda} = \frac{\partial \phi}{\partial \lambda}$

Note that $\frac{\partial \phi}{\partial \lambda}$ is taken as the partial derivative here (as opposed to a total derivative like $\frac{d \phi}{d \lambda}$), since we will see shortly that the form of our strain energy density of interest (Ogden) is a function of $\lambda$ rather than the strain invariants.

Also note that because $\lambda$ is diagonal, we are skipping the formal proof, here, for $\frac{\partial f(\lambda)}{\partial \lambda} = \frac{\partial f^2}{\partial \lambda} = 2 \lambda$ as well as the proof of $\frac{\partial \phi}{\partial \lambda} = \frac{\partial \phi}{\partial \lambda} I = \frac{\partial \phi}{\partial \lambda} \delta_{ij} e_i e_j$, along with the more obvious property of a diagonal tensor: $\lambda \cdot \lambda = \lambda^2 I = \lambda^2 \delta_{ij} e_i e_j$

With these substitutions, eq. (C.8) becomes

$$\sigma_i = \frac{2}{\lambda_1 \lambda_2 \lambda_3} \lambda_i^2 \left( \frac{1}{2 \lambda_i} \frac{\partial \phi}{\partial \lambda_i} \right) = \frac{1}{\lambda_j \lambda_k} \frac{\partial \phi}{\partial \lambda_i}$$  \hspace{1cm} (C.9)
C.4 Tabulated hyperelastic model

We start with the Ogden model - *viz*, eq. (6.19).

Let’s define a function:

\[
 f_0(\lambda) = \sum_{s=1}^{m} \mu_s \lambda^{s\alpha_s}
\]  

(C.10)

eq. (C.10) \(\rightarrow\) eq. (6.19) yields:

\[
 \sigma_i = \frac{1}{III} \left( f_0(\lambda_i) - \frac{1}{3} \sum_{n=1}^{3} f_0(\lambda_n) \right) + K \frac{III - 1}{III} \]  

(C.11)

Recall from the first Mooney-Rivlin example, that for an incompressible material under uniaxial test conditions, \(\lambda_j^* \approx \lambda_k^* \approx \lambda_i^{-1/2} \) (\(\lambda_j \approx \lambda_k \approx \lambda_i^{-1/2}\)), where the subscripts \(j\) and \(k\) refer to the two coordinate directions perpendicular to \(i\), just as before.

The engineering stress, which would be commonly retrieved from a uniaxial test, is the nominal stress for a hyperelastic material (recall eq. (4.5), which states that \(\sigma^0 = IIIF^{-1} \cdot \sigma\))

\[
 \sigma^0 = \lambda_i \lambda_j \lambda_k \begin{bmatrix} \lambda_i^{-1} & 0 & 0 \\ 0 & \lambda_i^{1/2} & 0 \\ 0 & 0 & \lambda_i^{1/2} \end{bmatrix} \begin{bmatrix} \sigma_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}
\]

\(\rightarrow\) \(\sigma^0_{11} = \lambda_j \lambda_k \sigma_{11}\)

No sum

eq. (C.11), for uniaxial load under the Ogden model, will be expressed as \(\sigma_1 = \sigma_{11} = "\sigma(\lambda_i)"\). So, \(\sigma(\lambda_i)\) is a particular value of longitudinal Cauchy stress under uniaxial loading, which corresponds to a particular value of the longitudinal stretch, \(\lambda_i\), under the Ogden model.

We’ll introduce a new variable, \(\epsilon_{0i}\), which is the engineering strain in a uniaxial test - *i.e.* \(\epsilon_{0i} = \lambda_i - 1\) for a hyperelastic material if \(\lambda_i\) is some longitudinal stretch that occurred during the uniaxial test. Presumably, we have an experimental curve of uniaxial engineering stress, which we will from now on call \(\sigma_0\), as a function of the longitudinal engineering strain (*i.e.* \(\sigma_0(\epsilon_{0i})\)).

With our new notation, we can define \(\sigma^0_{11}\) as follows:
C.4. TABULATED HYPERELASTIC MODEL

\[ \sigma_0(\epsilon_0) = \sigma_0(\lambda_i - 1) = \lambda_j \lambda_k \sigma(\lambda_i) \]

Now summing only on repeated indices unless otherwise noted, we can march through the derivation of this tabulated model. First, we observe eq. (C.10), and note that:

\[ f_0(\lambda_i) = \sum_{s=1}^{m} \mu_s \lambda_i^{* \alpha_s} \quad (C.12) \]

Additionally, we notice the term \( \frac{1}{3} \sum_{n=1}^{3} f_0(\lambda_n) \) in eq. (C.11), and are thus interested in the following calculation:

\[ \sum_{n=1}^{3} f_0(\lambda_n) = f_0(\lambda_i) + f_0(\lambda_j) + f_0(\lambda_k) = \left[ \sum_{s=1}^{m} \mu_s \lambda_i^{* \alpha_s} + 2 \sum_{s=1}^{m} \mu_s \lambda_i^{- \frac{\alpha_s}{2}} \right] \quad (C.13) \]

where \( f_0(\lambda_j) \) and \( f_0(\lambda_k) \) were taken to be \( f_0(\lambda_i^{-1/2}) \)

Substituting eq. (C.12) and eq. (C.13) into the original stress equation (e.g. eq. (C.11)) gives the following result:

\[ \sigma_0(\lambda_i - 1) = \lambda_j \lambda_k \sigma(\lambda_i) = \lambda_j \lambda_k \left( \frac{3}{2} f_0(\lambda_i) - \frac{2}{3} f_0(\lambda_i^{-1/2}) + K \frac{III_V - 1}{III_V} III_V \right) \]

\[ \lambda_i \sigma_0(\lambda_i - 1) + p = \frac{2}{3} f_0(\lambda_i) - \frac{2}{3} f_0(\lambda_i^{-1/2}) \quad (C.14) \]

\( III_V \) was eliminated from eq. (C.14) since we are going to limit our discussion to incompressible materials only.

Note that “p” is really a hydrostatic term that depends on “K,” which in our case is arbitrary. Simply striking the term would not stay true to the Ogden function and could cause undesirable behavior. However, we can eliminate the term through consideration of boundary conditions.
For uniaxial stress, \( \sigma(\lambda_j) = \sigma(\lambda_k) = \sigma(\lambda_i^{-1/2}) = 0 \).

Eq. (C.11) yields:

\[
0 = \frac{1}{3} f_0(\lambda_i^{-1/2}) - \frac{1}{3} f_0(\lambda_i) + \frac{K_{III}V - 1}{p} \tag{C.15}
\]

In eq. (C.15), we find that \( p \) must equal:

\[
p = \frac{1}{3} f_0(\lambda_i) - \frac{1}{3} f_0(\lambda_i^{-1/2}) \tag{C.16}
\]

Eq. (C.16) \( \rightarrow \) eq. (C.14) yields:

\[
\lambda_i \sigma_0 (\lambda_i - 1) = f_0(\lambda_i) - f_0(\lambda_i^{-1/2}) \tag{C.17}
\]

We can substitute consecutive values of the principal stretch into eq. (C.17).

\[\text{i.e.}\]

\[
\lambda_i^{-1/2} \sigma_0 (\lambda_i^{-1/2} - 1) = f_0(\lambda_i^{-1/2}) - f_0(\lambda_i^{1/4})
\]

\[
\lambda_i^{1/4} \sigma_0 (\lambda_i^{1/4} - 1) = f_0(\lambda_i^{1/4}) - f_0(\lambda_i^{-1/8})
\]

\[\text{etc.}\]

In general,

\[
\lambda_i^{(-1/2)x^{-1}} \sigma_0 \left( \lambda_i^{(-1/2)x^{-1}} - 1 \right) = f_0 \left( \lambda_i^{(-1/2)x^{-1}} \right) - f_0 \left( \lambda_i^{(-1/2)x} \right) \tag{C.18}
\]

Since \( \lim_{x \to \infty} f_0 \left( \lambda_i^{(-1/2)x} \right) = f_0(1) \), where \( f_0(1) = \sum_{s=1}^{m} \mu_s \),

we get:

\[
\sum_{x=1}^{\infty} \lambda_i^{(-1/2)x^{-1}} \sigma_0 \left( \lambda_i^{(-1/2)x^{-1}} - 1 \right) = f_0(\lambda_i) - f_0(1)
\]

where all terms on the right hand side cancel, except for the first and last.
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So, \( f_0(\lambda_i) = f_0(1) + \lambda_i \sigma_0(\lambda_i - 1) + \lambda_i^{-1/2} \sigma_0(\lambda_i^{-1/2} - 1) + \lambda_i^{1/4} \sigma_0(\lambda_i^{1/4} - 1) + \ldots \)

Writing this as concisely as possible:

\[
f_0(\lambda_i) = f(1) + \sum_{x=0}^{\infty} \lambda_i^{(-1/2)^x} \sigma_0 \left( \lambda_i^{(-1/2)^x} - 1 \right) \quad (C.19)
\]

We can now substitute eq. (C.19) into eq. (C.11). Since \( f(1) \) is a constant, we can see that it doesn’t affect the stress, \( \sigma_i \), since \( f(1) - 1/3 \sum_{n=1}^{3} f(1) = 0 \).

To complete our discussion, \( f_0(\lambda_i) \) and \( \sigma_0 \) will be written a final time, in their final form:

\[
f_0(\lambda_i) = \sum_{x=0}^{\infty} \lambda_i^{(-1/2)^x} \sigma_0 \left( \lambda_i^{(-1/2)^x} - 1 \right) \quad (C.20)
\]

\[
\sigma_i = \frac{1}{III_{\mathbf{V}}} \left( f_0(\lambda_i) - \frac{1}{3} \sum_{n=1}^{3} f_0(\lambda_n) \right) + K \frac{III_{\mathbf{V}} - 1}{III_{\mathbf{V}}} \quad (C.21)
\]

The way that this model works is described in section 6.2. The introduction of \( f_0(\lambda_i) \), which eliminates the material constants from the Ogden model (i.e. eq. (C.11)) was important, but it was the step from eq. (C.18) to eq. (C.19) that enables this “tabulated” method to work as desired. The particular pattern that was recognized by the aforementioned researchers that developed this “tabulated” method [17], which is expressed in eq. (C.18), along with the observation that summing the right-hand-side of eq. (C.18) cancels most of the terms, were really the key insights to isolate the \( f_0(\lambda_i) \) term.
Appendix D

Chapter 7 derivations

D.1 Jaumann rate in infinitesimal elasticity

Let’s define $\sigma$ as the infinitesimal stress tensor that we want to obtain, and $\hat{\sigma}$ as the infinitesimal stress tensor in material coordinates (here, $\sigma$ is in spatial coordinates, hence the use of the variable “$\sigma$” that has been previously reserved for the Cauchy stress). With these definitions, consider Fig. D.1.

![Diagram](image)

**Figure D.1: Rotating body without shear**

With respect to the stress measures depicted in Fig. D.1, let’s take the time derivative: i.e. Let us start with $\sigma_{ij}e_i e_j = \hat{\sigma}_{ij} \hat{e}_i \hat{e}_j$ since these two measures of stress are merely transformations of each other in linear infinitesimal elasticity, and are identical at time $t=0$. Now, take the time derivative of both sides of the equality.
\[ \dot{\sigma} = \dot{\sigma}_{ij} e_i e_j + \sigma_{ij} \dot{e}_i e_j + \sigma_{ij} e_i \dot{e}_j = \dot{\sigma}_{ij} \dot{e}_i \dot{e}_j + \dot{\sigma}_{ij} \dot{e}_i e_j + \dot{\sigma}_{ij} e_i \dot{e}_j, \]

where the slashed terms are zero. 

Also note: 

\[ \dot{\epsilon}_i = W_{ki} \dot{e}_k \]

\[ \rightarrow \sigma = \dot{\sigma}_{ij} \dot{e}_i \dot{e}_j + \sigma_{ij} W_{ki} \dot{e}_k \dot{e}_j + \sigma_{ij} \dot{e}_i W_{kj} \dot{e}_k = \dot{\sigma}_{ij} \dot{e}_i \dot{e}_j + \sigma_{kj} W_{ik} \dot{e}_i \dot{e}_j + \sigma_{ik} \dot{e}_i W_{jk} \dot{e}_j \]

\[ = \left[ \dot{\sigma}_{ij} + \sigma_{kj} W_{ik} + \sigma_{ik} W_{jk} \right] \dot{e}_i \dot{e}_j \]

\[ \dot{\sigma}_{ij} e_i e_j = \left[ \dot{\sigma}_{ij} + \sigma_{kj} W_{ik} + \sigma_{ik} W_{jk} \right] \dot{e}_i \dot{e}_j \] (D.1)

Recall that at the beginning of the derivation, we noted that \( \sigma_{ij} e_i e_j = \dot{\sigma}_{ij} \dot{e}_i \dot{e}_j \). In addition, note that \( \dot{\sigma}_{ij} \dot{e}_i \dot{e}_j \) needs to be transformed to the spatial bases, which can be accomplished using the transformation \( F \cdot \dot{\sigma} \cdot F^T \). Thus, eq. (D.1) becomes:

\[ \dot{\sigma} = F \cdot \dot{\sigma} \cdot F^T + \sigma \cdot W + W^T \cdot \sigma \] (D.2)

Recognizing \( F \cdot \dot{\sigma} \cdot F^T \) to be the Jaumann rate, with \( det F \) taken to be approximately unity for infinitesimal deformations, we arrive at the desired result:

\[ \dot{\sigma} = \sigma + \sigma \cdot W + W^T \cdot \sigma \] (D.3)

Eq. (D.3) is exactly the same as our previous expression for the Jaumann rate (eq. (7.13)). It is derived in a different, more general, way in Appendix D.2.
D.2 Truesdell and Jaumann rates

From eq. (4.6), we know that:

$$\sigma = \frac{1}{\det F} \hat{\sigma} \cdot F^T$$  \hspace{1cm} (D.4)

Before we take the time derivative of eq. (D.4), note that:

$$\frac{d}{dt} (\det F)^{-1} = - (\det F)^{-2} \cdot \dot{\det F} = -\frac{(\det F) \text{tr} D}{(\det F)^2}$$  \hspace{1cm} (D.5)

Note that in eq. (D.5), we used the equality $\dot{\det F} = (\det F) \text{tr} D$, which was proven in the derivation of the hypoelastic constitutive relationship involving the Jaumann rate (i.e. eq. (7.8)).

Now, taking the time derivative of eq. (D.4), we have:

$$\dot{\sigma} = -\frac{\text{tr} D}{\det F} \cdot \hat{\sigma} \cdot F^T + \frac{1}{\det F} \hat{F} \cdot \hat{\sigma} \cdot F^T + \frac{1}{\det F} \hat{F} \cdot \hat{\sigma} \cdot \dot{F}^T + \frac{1}{\det F} \hat{\sigma} \cdot \dot{F}^T$$

Now, we know $\hat{\sigma}$ in terms of $\sigma$ from eq. (4.6). Substituting, we get:

$$\dot{\sigma} = -\frac{\text{tr} D}{\det F} F \cdot (\det F)^{-1} \cdot \sigma \cdot \mathcal{F}^{-\mathcal{F}} \cdot \mathcal{F}^{-\mathcal{F}} + \frac{1}{\det F} \hat{F} \cdot (\det F)^{-1} \cdot \sigma \cdot \mathcal{F}^{-\mathcal{F}} \cdot \mathcal{F}^{-\mathcal{F}}$$

$$+ \frac{1}{\det F} \hat{F} \cdot \hat{\sigma} \cdot F^T + \frac{1}{\det F} \hat{F} \cdot (\det F)^{-1} \cdot \sigma \cdot \mathcal{F}^{-\mathcal{F}} \cdot \mathcal{F}^{-T} \cdot \hat{F}^T$$

Thus, we arrive at the desired result:

$$\dot{\sigma} = -\text{tr}(D)\sigma + L \cdot \sigma + \overline{\nabla} + \sigma \cdot L^T$$

Or,

$$\overline{\nabla} = \dot{\sigma} - L \cdot \sigma - \sigma \cdot L^T + \text{tr}(D)\sigma$$  \hspace{1cm} (D.6)

Throwing all $D$ terms out except for where it appears in the constitutive expression, we can arrive at the Jaumann expression (eq. (7.5), eq. (7.13), eq. (D.3)):

$$\dot{\sigma} = \dot{\sigma} + W \cdot \sigma + \sigma \cdot W^T$$  \hspace{1cm} (D.7)
Bibliography


